Semi-stable reduction and monodromy

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Abstract

The main theme of this Kleine AG is the reduction of varieties, more specifically curves and abelian varieties. The first aim is to prove the "Semi-Stable Reduction Theorem" for curves following the paper of Artin and Winters [1]. The second aim is to transfer this result to abelian varieties. The third aim is to put these results into the context of ℓ-adic monodromy in general.

1 Introduction

1.1 Monodromy and Models

Let R be a discrete valuation ring with field of fractions K and X/K a proper, smooth scheme over the generic point $\eta := \mathrm{Spec}(K)$ of $S := \mathrm{Spec}(R)$. Then X does not necessarily extend to a smooth proper scheme $\mathcal{X}$ over R. One possible obstruction is the existence of "monodromy" in the cohomology of X.

Let us illustrate this idea with the Legendre family of elliptic curves in an analytic setup. We consider

$$E_\lambda : \quad y^2 = x(x-1)(x-\lambda)$$

where $\lambda \in \Delta^*$ varies in the punctured unit disk. For $\lambda = 0$, this equation degenerates to $y^2 = x^2(x-1)$ which is a nodal curve and hence singular. But even with other equations for E_λ, it is impossible to find a smooth proper continuation of this family to $\Delta = \Delta^* \cup \{0\}$.

Namely, the family $f : E_\lambda \to \Delta^*$ is topologically a fibre bundle and so $R^1 f_* \mathbb{Z}$ is a locally free sheaf with stalk $\mathbb{Z}^2$. After fixing a base point λ_0, this sheaf corresponds to a representation $\rho : \pi_1(\Delta^*, \lambda_0) \to GL(H^1(E_{\lambda_0}, \mathbb{Z}))$, called the monodromy representation for E_λ. In our situation, ρ turns out to be nontrivial and hence cannot factor over $\pi_1(\Delta, \lambda_0) = \{1\}$. In particular, it cannot arise by restriction from a smooth proper family over the full disk Δ. We could perform the same arguments with the $\mathbb{Z}/\ell^n\mathbb{Z}$-local system $E_\lambda[\ell^n]$, which is also nontrivial.

This last point of view translates to the purely algebraic setting. For this we replace $\Delta, \Delta^*, \pi_1(\Delta^*, \lambda_0)$ by S, its generic point η and the inertia group I of $R \subset K$. Then we consider an abelian variety A/K together with its Tate-module $T_\ell(A)$, which is equipped with a $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$-action. In this situation the criterion of Néron-Ogg-Shavarevich states that A has good reduction if and only if I operates trivially on $T_\ell(A)$.

Now note that $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(\mathrm{A}), \mathbb{Q}_\ell) \cong \mathrm{H}^1(\mathrm{A}_{\overline{\mathrm{K}}}, \mathbb{Q}_\ell)^\vee$ and $H^*_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_\ell) = \bigwedge^* H^1_{\mathrm{et}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ as Galois modules. So we see that the whole cohomology of A is unramified if A has good reduction. This generalizes to an arbitrary smooth proper variety X/K as follows. If X has a smooth proper model over R (i.e. if X has good reduction), then its ℓ-adic cohomology is unramified. (Talk 5)

1.2 The Semi-stable Reduction Theorem

Of course, it is often possible to obtain *some* proper flat model of X over S. For example any embedding $i : X \hookrightarrow \mathbb{P}^n_K$ yields such a model $\mathcal{X}/S$ by taking the closure $\mathcal{X} = \overline{i(X)} \hookrightarrow \mathbb{P}^n_R$. But the special fibre $\mathcal{X}_s$ can be very singular, depending on the chosen embedding i.

Now let X/K be a curve of genus g. Then the semi-stable reduction theorem (Talk 1-3) ensures the existence of a reasonable model for X in the following sense.

Definition 1.1. Let k be some field with algebraic closure $\bar{k}$. A proper curve C/k is called semi-stable if it is geometrically connected, geometrically reduced and satisfies:

- All singular points of $C_{\bar{k}}$ are ordinary double points, i.e. the completions of their local rings are isomorphic to $\bar{k}[[x, y]]/(xy)$.
- Every non-singular rational irreducible component of $C_{\overline{k}}$ meets at least 2 other components.

Theorem 1.2 (Semi-Stable Reduction Theorem for curves)**.** *Let R be a discrete valuation ring with field of fractions K. Assume that X over K is a connected proper smooth curve of genus $g \geq 2$. Then there exists a finite extension K' of K and a proper, flat model $\mathcal{X}$ of $X_{K'}$ over the integral closure R' of R in K' with all fibers being semi-stable curves.*

The proof given in [1] is based on two things. First, a careful study of the "combinatorics" of possible special fibers of regular models for X. Second, an understanding of the relation between the geometry of the curve and the group structure of its Jacobian. To illustrate the theorem we sketch the cases $g = 0$ and $g = 1$:

If $g = 0$, we can assume that $X \cong \mathbb{P}^1_K$ after an extension of K. Then we choose $\mathbb{P}^1_R$ as model.

In the case $g = 1$, we assume for simplicity that the residue field of K is not of characteristic 2 (see [5, VII.5.4]). Then we can extend K so that X is given in Legendre form

$$X : \quad y^2 = x(x-1)(x-\lambda)$$

with some $\lambda \in K$. If $\lambda \in R$, then this equation yields a semi-stable model of X. Otherwise we choose a uniformizer $\pi \in R$ and $r \in \mathbb{N}$ such that $\pi^r \lambda \in R^*$ is a unit. Applying the coordinate change $x' = \pi^r x, y' = \pi^{3r/2} y$ over $K(\sqrt{\pi})$ yields an equation

$$X : \quad (y')^2 = x'(x' - \pi^r)(x' - \pi^r \lambda)$$

with integral coefficients and semi-stable reduction.

Talk 1 - Finiteness of the number of fiber types

This talk is mainly of combinatorial nature, but will prove a crucial ingredient for the proof of the semi-stable reduction theorem, namely theorem 1.16 of [1]. It should cover all of [1, §1]. More precisely, the following should be presented:

- basic properties of special fibers of regular models over DVR's for smooth curves [1, (1.1)] culminating in the definition of a "type"[1, (1.2)] and its "genus" [1, (1.3)]
- properties of types [1, (1.4)] and a sketch of proof for [1, (1.6)]
- the definitions of the group $G = G(T)$ for a type T [1, (1.10)] and the numbers $\rho_c(G)$
- the remaining statements [1, (1.13)-(1.19)] with as many details and examples as possible, especially for theorem (1.16).

The following statement deduced from [1, (1.16)] will appear in [1, (2.8)]: With the notations in theorem [1, (1.16)] there exists an integer N only depending on g such that for every prime $s > N$ the dimension over $\mathbb{Z}/s$ of the s-torsion $G[s]$ is at most β. A proof for this claim should be given in the end of this talk.

Talk 2 - Results on the Picard functor for curves

In contrast to the previous talk this talk is mainly "geometric". It will discuss the Picard functor of general proper curves over fields. The main reference is [2, 9.2], but the main results for this talk can also be found in [4, §2]. Here is a detailed list for this talk:

- Start with the definition of the Picard functor $\mathrm{Pic}_{X/S}$ [2, 8.1.2] for a morphism $f : X \longrightarrow S$ and show that $\mathrm{Pic}_{X/S} \cong R^1 f_*(\mathbb{G}_m)$ (directly after the definition). For simplicity (and because for the semi-stable reduction theorem finite base extensions are allowed anyway) it can be assumed that f admits a section.
- State the representability theorems for the Picard functor [2, 8.2.1, 8.2.3] and shortly discuss its local properties [2, 8.4.2] to deduce that its representing scheme is smooth for proper curves over fields.
- Present proposition 3,5,9,10 and example 8 of [2, 9.2] finally reaching [2, 9.2.12] resp. [4, §2.(2.6)].
- Prove [1, §2.(2.1)].

Talk 3 - Existence of semi-stable reduction

Finally, we will arrive at our main theorem [1, (2.10)] about semi-stable reduction of curves. This talk will present its proof benefiting from the work done in the previous two talks. As [1, (2.1)] should have been proven in talk 3, this talk starts with [1, (2.3)] and should cover the rest of [1, §2] without the last remark (which is part of talk 4). Building upon the results of talk two and three it should be possible to give detailed proofs of all occurring statements.

Talk 4 - Semi-abelian reduction for Abelian Varieties

This talk will explain the semi-abelian reduction theorem for abelian varieties following [2]. Namely every abelian variety has semi-abelian reduction. The criterion of Néron-Ogg-Shavarevich relates this theorem to l-adic Galois representations.

- Recall the definition of the Néron model of an abelian variety [2, Definition 1, §1.2] and state the main existence theorem [2, Corollary 2, §1.3].
- Recall the definition of abelian and semi-abelian reduction as in [2, §7.4] and explain the proof of loc. cit., Theorem 1.
- Prove the equivalence of a) and d) in [2, Theorem 5, §7.4].
- Finally mention [2, Theorem 6, §7.4].

Talk 5 - Cohomology of varieties with good reduction

In this talk we prove that ℓ-adic Galois representations of varieties with good reduction are unramified. This talk requires some knowledge of étale cohomology and it has to be a bit sketchy.

- Recall the relation of the Tate module of an abelian variety to its étale cohomology.
- Prove that the ℓ-adic cohomology of a variety with good reduction is unramified. This is equivalent to [3, Theorem V 3.1] in the case of a strictly henselian base.

References

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