Kleine AG: Arakelov geometry and the Bogomologov conjecture

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Introduction

The goal of this Kleine AG is the proof of the Bogomolov conjecture:

Theorem 1 (Bogomolov Conjecture)**.** *Let X be a smooth algebraic curve over a number field K of genus $g \geq 2$, $\overline{K}$ the algebraic closure of K and fix an embedding $X \to J$ of X into its Jacobian J. Denote by h the Neron-Tate height on J. Then there exists an $\epsilon > 0$ such that $\{x \in X(\overline{K}) \mid h(x) \leq \epsilon\}$ is finite.*

Since the points $P \in J(\overline{K})$ with $h(P) = 0$ are exactly the torsion points of J, we get as a corollary the Manin-Mumford conjecture:

Corollary 2 (Manin-Mumford Conjecture)**.** *The intersection $X(\overline{K}) \cap \mathrm{Tors}(\mathrm{J}(\overline{\mathrm{K}}))$ of the geometric points of X and the torsion points of J is finite.*

We want to prove the Bogomolov conjecture by using Arakelov geometry. Our main reference is [KMY]. We sketch shortly the main ideas of Arakelov geometry:
For algebraic curves over a function field F one can use stable reduction theory to get an complete algebraic surface over a curve, such that F is the function field of this curve. Here, one can use intersection theory of divisors for proving many interesting theorems (e.g. the Mordell conjecture). In the same way we can extend a smooth curve X over a number field K of genus g to a curve $\mathcal{X}$ over $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ denotes the integers of K. We call $\mathcal{X}$ an arithmetic surface. Since the base is no longer complete as in the function field case, we can not expect a nice intersection theory in the same way, e.g. there is no moving lemma. But Arakelov introduced 1974 in his paper [Ar] another divisor group. If we denote by S_∞ the set of infinite places of K, an *Arakelov Divisor* has the form:

$$D = D_f + D_\infty,$$

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where D_f an usual Weil divisor on X and $D_\infty = \sum_{w \in S_\infty} r_w F_w$ is a formal linear combination with $r_w \in \mathbb{R}$ and F_w is just a symbol meaning "the fibre of X over the infinite place v". If we consider $w \in S_\infty$ as an embedding $K \xrightarrow{w} \mathbb{C}$, we denote by X_w the Riemann surface corresponding to the base change of X by w. If we choose an ON-basis $\omega_1, \ldots, \omega_g \in \Gamma(X_w, \Omega^1_{X_w})$ for the scalar product $\langle \omega, \omega' \rangle = \frac{i}{2} \int_{X_w} \omega \wedge \overline{\omega'}$, we get the canonical form $\mu_w = \frac{i}{2g} \sum_{j=1}^g \omega_j \wedge \overline{\omega_j}$. For a rational function $f \in K(\mathcal{X})$ we now define the corresponding principal Arakelov divisor by

$$\mathrm{div}_{\mathrm{Ar}}(f) = \mathrm{div}(f) + \sum_{w \in S_\infty} \left(\epsilon_w \int_{X_w} - \log |f|_w \mu_w \right) F_w,$$

where ϵ_w is given by

$$\epsilon_w = \begin{cases} 1 & \text{for } k_w = \mathbb{R}, \\ 2 & \text{for } k_w = \mathbb{C}. \end{cases}$$

We define $\mathrm{Ch}_{\mathrm{Ar}}(\mathcal{X}) = \{\text{Arakelov Divisors}\} / \{\mathrm{div}_{\mathrm{Ar}}(f) \mid f \in K(\mathcal{X})\}$ to be the *Arakelov-Chow group*. Arakelov has shown, that there is an intersection product on $\mathrm{Ch}_{\mathrm{Ar}}(X)$ in form of a formal sum $\langle D, E \rangle = \sum_{w \in S} \langle D, E \rangle_w$, where $S = S_f \cup S_\infty$ is the set of finite and infinite primes of K. If D and E are usual Weil divisors and w is finite, $\langle D, E \rangle_w$ is some multiple of the usual intersection multiplicity of D and E over the point $w \in \mathrm{Spec}(\mathcal{O}_K)$. Hence, in some sense it extends the usual intersection product.

There is an alternative description of $\mathrm{Ch}_{\mathrm{Ar}}(\mathcal{X})$ by *metrized line bundles*. A metrized line bundle is a line bundle $\mathscr{L}$ on $\mathcal{X}$ together with a metric ρ_w on $\Gamma(\mathcal{X}, \mathscr{L}) \otimes K_w$ for each infinite place $w \in S_\infty$. For a point $P \in \mathcal{X}$ we have a canonical metric on the line bundle $\mathcal{O}_{\mathcal{X}}(P)$ given by $\log \|1(Q)\|_{\rho_w} = -\epsilon_w \cdot g(P, Q)$, where 1 means the constant section 1, $Q \neq P$ is another point and g is the Arakelov-Green function, which is unique by the following properties:

(i) If z is a local coordinate at P with $z(P) = 0$, we have locally $g(P, Q) = \log |z(Q)| +$ C^∞-function.

(ii) $\delta_P \overline{\delta_P} g(P, Q) = -\frac{\pi}{2g} \sum_{j=1}^g \omega_j \wedge \overline{\omega_j}$

(iii) $\int_{X_w} g(P, Q) \mu_w(P) = 0$

Now we get a canonical metric on every $\mathcal{O}_{\mathcal{X}}(D_f)$ for a Weil divisor D_f, because every $\mathcal{O}_{\mathcal{X}}(D_f)$ is a tensor product of $\mathcal{O}_X(P)$'s. For an Arakelov divisor $D = D_f + \sum_w r_w \cdot F_w$ we define $\mathcal{O}_{\mathcal{X}}(D)$ to be the line bundle $\mathcal{O}_{\mathcal{X}}(D_f)$ with the canonical metric multiplied by e^{-r_w}. We call a metrized line bundle *admissible*, if it comes from a Arakelov divisor and we define the *Arakelov-Picard group* $\mathrm{Pic}_{\mathrm{Ar}}(\mathcal{X})$ to be the group of isomorphism classes of admissible metrized line bundles. We have a natural isomorphism

$$\mathrm{Ch}_{\mathrm{Ar}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Pic}_{\mathrm{Ar}}(\mathcal{X}), \ D \mapsto \mathcal{O}_{\mathcal{X}}(D).$$

Now one can define interesting and useful invariants like the self-intersection number of the canonical sheaf or height functions as the degree of the intersection product of a closed point and a fixed line bundle. For more details on Arakelov geometry of arithmetic surfaces we recommend the very well readable book [La] by Lang.

However we want to apply Arakelov geometry to abelian varieties, hence we have to study it also in higher dimensions. The higher dimensional Arakelov geometry is due to Gillet and Soulé (see [SABK]) and unfortunately it is much more complicated. In the first talk we study the higher dimensional analogue of the upper constructions. The second talk shows, that under certain conditions there are small sections for metrized line bundles, i.e. sections with $\|s\| < 1$. This is very useful, since the Arakelov definition of the degree contains a summand $\int_{X(\mathbb{C})} -\log \|s\| \dots$, but we are often only interested in lower bounds. The crucial step, where Arakelov theory comes in for the proof of the Bogomolov conjecture, is that the Neron-Tate height function is equal to a certain height function coming from Arakelov geometry. To construct this height function, we have to construct the corresponding (admissible) metric, which is done in the third talk. The next talk treats arithmetic height functions, which can be considered as the degree of the intersection of a fixed line bundle with a closed point. The main statement of this talk are two important inequalities between the height function and the self-intersection number of the fixed line bundle. The last talk proves the Bogomolov conjecture by introducing the equidistribution theorem. If we choose a generic sequence of closed points x_m, such that the height of x_m goes to zero for $m \to \infty$, the equidistribution theorem asserts that the limit of certain currents depending on x_m under some conditions is independent of the sequence and can be given in terms of the first chern class of the fixed line bundle. This theorem is used then to construct a contradiction, if such a sequence in an abelian variety does not lie in a translation of a subvariety by a torsion point, which proves the Bogomolov conjecture.

Every talk should take 45 minutes and for all talks the reference is [KMY].

Talk 1: The arithmetic Chow ring

In this talk we introduce the basic notions, extending their well-known definitions in the case of varieties to the arithmetic setting. We define arithmetic schemes $\mathcal{X}$ over Spec $\mathbb{Z}$ and hermitian line bundles $(\mathcal{L}, h)$ on it (where h is a hermitian metric on the complexification of $\mathcal{L}$). To be able to define the arithmetic Chow group, we need (Green) currents on compact complex manifolds X, i.e. linear forms on the space of (p, q)-forms on X. The arithmetic Chow group $\widehat{CH}^p(\mathcal{X})$ on an arithmetic scheme $\mathcal{X}$ is then the set of pairs consisting of a cycle $\mathcal{Z} \subset \mathcal{X}$ (of codimension p) and a Green current g on its complexification, taken up to rational equivalence. Finally we state the existence of an intersection product

$$\widehat{CH}^p(\mathcal{X}) \times \widehat{CH}^q(\mathcal{X}) \to \widehat{CH}^{p+q}(\mathcal{X}) \otimes \mathbb{Q}$$

and give some properties concerning pull-back and push-forward. This talk should cover sections 1.2-1.4.

Apart from the definitions of the objects already mentioned above, we will need the following statements:
Proposition 1.2.11 on the existence of Green currents for cycles shows that the definition of the arithmetic Chow group is sensible. The arithmetic first Chern class (on p.12) will play an important role. Cover the map 1.4.4.0 on p.15, which is needed later on to make sense of $\widehat{c}_1(\mathcal{L})^{d+1}$, and Proposition 1.3.4 as well. If time permits it would be good to see the connection between hermitian metrics on line bundles and Green currents on the

corresponding divisor, i.e. Example 1.2.7 (together with the statement of Theorem 1.2.8) and Proposition 1.2.14.

Talk 2: Small sections

Consider a hermitian line bundle $(\mathcal{L}, h)$ on an arithmetic scheme $\mathcal{X}$ inducing in the usual fashion a norm $\|.\|$ on $\mathcal{L}$. Then a small section is an element $s \in H^0(\mathcal{X}, \mathcal{L})$ such that $\|s\|_{sup} = \sup\{\|s\|(x) \mid x \in \mathcal{X}(\mathbb{C})\} < 1$. The main result of this talk is, that under certain conditions such small sections always exist. On the way towards this statement we will encounter the arithmetic Euler characteristic and an arithmetic version of the Hilbert-Samuel theorem. This talk should focus on sections 3.1 and 3.3.

More precisely, we suggest explaining the following:
Before stating the Hilbert-Samuel theorem, you need to define the arithmetic Euler characteristic χ_{sup} as on p.28 (requiring selected parts of section 3.2), the degree map $\widehat{\deg} : \widehat{CH}^{d+1} \to \mathbb{R}$ as in section 1.6 (it should suffice to define it in the case of regular schemes and just note that one may extend it to all arithmetic schemes) and *vertically ample* resp. *nef* line bundles as in the beginning of section 3.3. Instead of proving the arithmetic Hilbert-Samuel Theorem 3.3.1 in general, give (in as much detail as possible) an overview of the proof of Theorem 3.3.3, i.e. under the additional hypothesis that $\mathcal{X}_{\mathbb{Q}}$ is regular and the line bundle is vertically ample. Apart from the top of p.30, the equations on the bottom of p.35 are of key importance for this. Finally state and prove (in full) the existence of small sections, i.e. Corollary 3.3.2.

Talk 3: Adelic and admissible metrics

In this talk we define adelic metrics on line bundles over varieties X over number fields K. These consist roughly speaking of a metric for each place of K, satisfying compatibility conditions concerning integral models of X. We see that we can extend the definition of $\widehat{\deg}(\widehat{c}_1(\mathcal{L}^{d+1}))$ to line bundles with adelic metrics over X. For the rest of this talk we restrict ourselves to the case of abelian varieties. There one can see that by successively pulling back an adelic metric along the n-times morphism gives in the limit a very special metric, which we call an admissible metric. We see that such an admissible metric is always a cubic metric, a fact that will become important later on. Everything in this talk is contained in section 4.

More precisely, we suggest explaining the following:
Out of section 4.1. cover the definition of a metric, a bounded and continuous metric and an adelic metric (Definition 4.1.2). Give Definition 4.1.3 and state Theorem 4.1.5. Out of section 4.2. (which we will only need in the case of X an abelian variety), give Theorem 4.2.1 which defines admissible metrics $\|.\|_0$ and Definition 4.2.3 of a cubic metric. Focus on Proposition 4.2.3 with proof of part (1) and stress the point that the proof shows that the admissible metric coming from multiplication by 2 (or more general n) defines a cubic metric.
If time permits define vertically ample/nef adelic metrics (Definition 4.1.4) and show that the admissible metric on an ample line bundle coming from multiplication by 2 is vertically nef (cf. page 48) and $c_1(\mathcal{L})$ is a positive definite C^∞-form (cf. second half of page 50).

Talk 4: Arithmetic height functions

We define the height function $h_{(\mathcal{X},\mathcal{L})}$ on a arithmetic scheme (or its generalization to any number field) coming with a hermitian line bundle. Moreover the height function defined for two models of one scheme over the generic fiber differ only by some bounded function. Nevertheless in the case of an abelian variety over a number field, we may in fact single out a specific one, the Néron-Tate height function. We will see in the next talk that this height function corresponds to the admissible metric defined by multiplication by 2. Finally we show the following connection between height functions and the intersection number:

$$\sup_{Y\subset\mathcal{X}_{\mathbb{Q}}}\left\{\inf_{x\in(\mathcal{X}_{\mathbb{Q}}\setminus Y)(\overline{\mathbb{Q}})} h_{(\mathcal{X},\mathcal{L})}(x)\right\} \geq \frac{\widehat{\deg}(\widehat{c}_1(\mathcal{L})^{d+1})}{(d+1)\deg(\mathcal{L}_{\mathbb{Q}}^{d+1})} \geq \inf_{x\in\mathcal{X}_{\mathbb{Q}}(\overline{\mathbb{Q}})} h_{(\mathcal{X},\mathcal{L})}(x)$$

where $d=\dim\mathcal{X}$. This talk covers most of section 5 (except 5.3).

More precisely, we suggest explaining the following:
Give the definition of the height function on p.51, state the independence from integral models (up to bounded functions) (cf. bottom of p.52) and give Theorem 5.1.4. Define the Néron-Tate height function as in section 5.2 and give Proposition 5.2.2. Give now Theorem 5.4.1 and Theorem 5.5.1 with proving as much as you can. Stress the important role, that small sections play in the proof of Theorem 5.4.1.

Talk 5: The Bogomolov conjecture

This talk combines all the results we obtained so far and finally proves the Bogomolov conjecture. Note that the assertion of Theorem 6.2.1, which we will prove, is a generalization of our Thereom 1 in the introduction, since a curve of genus $g\geq 2$ is never a translation of a subvariety by a torison point. This talk is essentially contained in section 6.

More precisely, we suggest explaining the following:
State theorem 5.3.1 (without proof). It has two corollaries which should be mentioned: First of all it follows finally that the Néron-Tate height function corresponds to the admissible metric defined by multiplication by 2[1]. The second (and more immediate) Corollary of 5.3.1 (and 5.5.1) is Corollary 5.5.3 (whose proof can be omitted as well). The rest (and vast majority) of the talk is to cover section 6 in as much detail as possible. Final state the Manin-Mumford conjecture (Corollary 2), which follows directly from the Bogomolov Conjecture by Proposition 5.2.2 (2).

[1]But as the easy proof is nowhere given explicitly, we add it here for completeness: From Theorem 5.3.1 (and 5.1.3) it follows that the height functions defined by the admissible metric differs only by a bounded function from the one defined by the original metric and hence the same is true wrt. the Néron-Tate height function. Moreover the height function for the admissible metric and the Néron-Tate height function satisfy

$$h_L(x+y+z)-h_L(x+y)-h_L(y+z)-h_L(z+x)+h_L(x)+h_L(y)+h_L(z)=0$$

Hence they are equal.

References

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