KLEINE AG
ANABELIAN GEOMETRY: NEUKIRCH-UCHIDA AND TAMAGAWA

Organization:
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The goal of this *Kleine AG* is to understand the anabelian results of Neukirch-Uchida and Tamagawa. We notice that a session of the *Kleine AG* with a very related topic took place 18 years ago. Its program [Fi] may contain further information, to which we refer the interested participants. Roughly speaking anabelian geometry is about reconstructing information about a scheme (e. g. its rational points, discrete invariants or even the scheme itself) from its étale fundamental group, or rather from the attached outer Galois representation, if one is in the relative situation of a scheme defined over a field. The "anabelian" program originates in a letter of Grothendieck to Faltings written 1983 [Gr]. The general anabelian philosophy is that there should exist an appropriate "anabelian" category of schemes (e. g. over a fixed base field), such that the étale fundamental group of any scheme in this category contains a lot information on the geometry of the scheme and moreover, determines the scheme uniquely. The word "anabelian" should indicate the fact that the involved fundamental groups are quite complicate and in a sense very far from being abelian.

Among other things, Grothendieck conjectured in [Gr] that hyperbolic curves (i. e. curves with negative Euler characteristic) can be recovered from their étale fundamental group. We will be concerned with the birational analog of this for number fields resp. function fields (this is the theorem of Neukirch-Uchida [Ne, Uc]) and with the proof of Grothendieck Conjecture for *affine* hyperbolic curves, due to Tamagawa [Ta]. Here the restriction to affine curves is essential, as the projective case turns out to be much harder and was proven by Mochizuki in [Mo]. Before coming to the talks, let us state the theorem of Neukirch-Uchida.

Theorem (Neukirch, Uchida). *Let K_1 and K_2 be two global fields. Let G_{K_1}, G_{K_2} be their absolute Galois groups with respect to some chosen separable closures. Then the natural map of sets*

$$\phi_{K_1,K_2} \colon \operatorname{Isom}(K_2, K_1) \longrightarrow \operatorname{Isom}(\mathrm{G}_{K_1}, \mathrm{G}_{K_2})/\operatorname{Inn}(\mathrm{G}_{K_2})$$

is bijective.

On the right hand side we consider isomorphisms of topological groups modulo the action of G_{K_2} given by composing an isomorphism with an inner automorphism of G_{K_2}. The map in the theorem is defined as follows: let $\alpha\colon K_2 \longrightarrow K_1$ be an isomorphism. Extend α to an isomorphism $\overline{\alpha}\colon K_2^{\mathrm{sep}} \longrightarrow K_1^{\mathrm{sep}}$ of separable closures. Then $\phi_{K_1,K_2}(\alpha)$ corresponds to the element $\sigma \mapsto \overline{\alpha}^{-1}\sigma\overline{\alpha}$ of $\operatorname{Isom}(\mathrm{G}_{K_1}, \mathrm{G}_{K_2})$. As differences by inner automorphisms of G_{K_2} become trivial, this does not depend on the choice of the lift $\overline{\alpha}$.

First talk. Introduction to anabelian geometry. (60 minutes) The aim of this overview talk is to give an introduction to the anabelian program of Grothendieck following [Gr]. Attention should be given to explaining the exact sequence

$$1 \longrightarrow \pi_1(\overline{X}_{\text{ét}}) \longrightarrow \pi_1(X_{\text{ét}}) \longrightarrow \mathrm{G}_k \longrightarrow 1$$

attached to a scheme X over a field k, where $\overline{X} = X \times_k k^{\mathrm{sep}}$, and to the corresponding outer Galois representation (a short but concise presentation can be found in [St, 4.1.1]). Explain the

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Isom-conjecture and the Section Conjecture, which should be satisfied for a class of *anabelian* varieties. Define hyperbolic curves as an example of anabelian varieties. Give some simple examples of non-anabelian varieties. As an example of an anabelian result in the zero-dimensional case, state the theorem of Neukirch-Uchida [NSW, 12.2.1] and mention the generalization by Pop [Po, Po2, Po3], which shows that the result is true in the much more general setup of infinite fields finitely generated over their prime field. In the one-dimensional case state the results of Tamagawa [Ta, theorems (0.3), (0.5)].

Second talk. Neukirch-Uchida theorem I. (45 minutes) The first part of this talk should concern anabelian properties of local fields (cf. [NSW, discussion after 12.1.7] and [Iv, section 2.2]). Explain which invariants of a local field κ can be recovered from its absolute Galois group G_κ and how this can be done. Finally, mention that local fields are not anabelian (cf. [NSW, remark after 12.2.7 and discussion before 7.5.15]) and sketch some arguments.

The second part of this talk should be concerned with the question, of how decomposition groups of various primes of a global field k lie in the absolute Galois group of k, cf. [NSW, 12.1.1-12.1.7].

Third talk. Neukirch-Uchida theorem II. (60 minutes) The goal of this talk is to explain Neukirch's proof of the Local Correspondence [NSW, 12.2.4], which almost immediately follows from the crucial result of Neukirch [NSW, 12.1.9]. From the Local Correspondence one easily deduces the following part of the Neukirch-Uchida theorem: if K_1, K_2 are number fields such that $K_1/\mathbb{Q}$ is normal and $\mathrm{G}_{K_1} \cong \mathrm{G}_{K_2}$, then $K_1 \cong K_2$.

Further, the speaker should decide how much time he wants to devote to the following two topics: (i) the Hasse principle and the Grunwald-Wang theorem (cf. [NSW, Chap. IX §1,2]) which are necessary for the proof of [NSW, 12.1.9] and (ii) the proof of the full Neukirch-Uchida theorem [NSW, 12.2.1]. At the very least sketch how Čebotarev density arguments play an important role and the major steps of the proof of the full Neukirch-Uchida theorem.

Fourth talk. Anabelian properties of affine hyperbolic curves over finite fields. (60 minutes) This talk is devoted to Tamagawa's proof of Grothendieck's Conjecture for affine hyperbolic curves. One should try to cover as much as possible of Sections 1-4 of [Ta], the "main" result being Proposition (3.8) (cf. Proposition (0.7)). One should restrict to the case of curves over finite fields to save time and try to avoid technical details wherever it is possible.

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