Kleine AG: Néron smoothening

Organisation:
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The main result of this Kleine AG is the following theorem, which (in a rather different language [3]) first appeared in [Nér]:

Main Theorem 1. *Let R be a DVR and X a scheme of finite type over R, whose generic fiber X_η is smooth. Then there exists a smoothening $f : X' \to X$ of X, i.e. a proper morphism f, s.th.*

- *f is an isomorphism on generic fibers*
- *For every étale R-algebra R', the canonical map $X'_{sm}(R') \to X(R')$ is a bijection. Here $X'_{sm} \subset X'$ denotes the smooth locus of X'.*

In order to prove this theorem, we will introduce Néron's measure of the defect of smoothness and see that we may reduce this defect by blow-ups in the special fiber.
This theorem was originally used to construct Néron models of arbitrary abelian varieties. Nevertheless it hardly possible to even sketch this proof in just one talk. To give an application, we will therefore focus on Artin's approximation theorem instead:

Theorem 2. *Let R be a field or an excellent DVR. Let A be the henselization of a R-algebra A_0, which is essentially of finite type over R. Let $\mathfrak{m}$ be any ideal of A and $\widehat{A}$ the $\mathfrak{m}$-adic completion of A. Fix any finite set of polynomials $f_1, \ldots, f_r \in A[Y_1, \ldots, Y_n]$ and a common zero $(\widehat{y_1}, \ldots, \widehat{y_n}) \in \widehat{A}^n$ of all the f_i. Then for all positive integers $c > 0$ there exists a common zero $(y_1, \ldots, y_n) \in A^n$ of all f_i such that for all $j = 1, \ldots, n$:*

$$y_j \equiv \widehat{y_j} \bmod \mathfrak{m}^c.$$

Note that it is conjectured (cf. [Art]), that the statement remains true for all excellent henselian rings A. This was shown in [Rot] under the additional assumption, that A contains $\mathbb{Q}$. Nevertheless time does not even permit us to give a full prove of the theorem as stated above.
The main reference for all talks will be [BLR], but you may also have a look at [Nér] if you feel up to a challenge. If no reference is specified, it always points to [BLR].

1. Statement and Dilatations (45 minutes)

The aim of this talk is to explain the statement of the main theorem and to define dilatations as open subschemes of blow-ups in the special fiber, as presented in [BLR, chap. 3.1 and 3.2].
More precisely, you should cover:

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[3]and we do not talk about the difference between French and English here!

- State definition 3.1/1 (of a smoothening) and the Main Theorem 3.1/3.
- Cover the comments on definition 3.1/1, i.e.
  - Give a brief reminder on strict henselizations of rings (and in particular of DVRs). Explain the reformulation of the second condition in the Main Theorem in terms of $X(R^{sh})$.
  - Explain (using hand-waving, not proofs) that a resolution of singularities is always a smoothening, but not conversely.
- State and prove proposition 3.2/1, which introduces dilatations.
- Explain (in as much detail as time permits) the example given at the beginning of chap. 3.2. As this also motivates the proof, you may consider giving the example right after stating proposition 3.2/1.
- State proposition 3.2/2 and 3.2/3 (without proofs, except if you have some time left).

2. Néron's measure of the defect of smoothness (45 minutes)

We will introduce Néron's measure of the defect of smoothness as the length of the torsion part in the sheaf of differential forms (at a given point). Then we prove some basic properties. This talk covers essentially the first half of [BLR, chap. 3.3].
More precisely, this talk should contain:

- Give a brief overview of the basic ideas to prove the Main Theorem (in the flavor of [BLR, p.67] (directly after the proof of prop. 3.3/3)).
- Define Néron's measure at points in $X(R^{sh})$.
- Cover lemma 3.3/1, lemma 3.3/2 and proposition 3.3/3 (with proofs). Make sure that the audience gets some feeling about this measure. You may even consider skipping the proofs of the lemmas in order to give some explicit examples instead.
- Define the condition (N) and state lemma 3.3/4. You may omit the motivation for this definition (as this will become clear in the next talk).

3. Lowering the defect of smoothness (45 minutes)

We show now that blowing up in some (sufficiently nice) closed subscheme $Y_k \subset X_k$ in the special fiber, reduces the defect of smoothness for all points which specialize into the smooth locus of Y_k. This key result is the content of [BLR, prop. 3.3/5].
It should be clear, what to do in this talk. So you are fine, if you...

- State prop. 3.3/5 and give an example (e.g. the one following directly afterwards)
- ... and most importantly explain the proof of prop. 3.3/5

4. Finishing the Proof (30 minutes)

We are now ready to prove the Main Theorem as done in [BLR, chap. 3.4]. We suggest to cover the following topics:

- Give the definition of E-permissibility.
- You may consider to state (as a reminder) lemma 3.4/1 and theorem 3.4/2, which (almost) coincide with prop. 3.3/5 resp. our Main Theorem.
- Finally give the proof of the Main Theorem, i.e. the proof of thm. 3.4/2.

5. Artin's approximation theorem (45-60 minutes)

We sketch the proof of Artin's approximation theorem, which was already stated in the introduction above. This is contained in [BLR, chap. 3.6].
Summarize chapter 3.6 as you see fit. As a guideline you may consider to

- state that the second condition in the Main Theorem stays true for arbitrary flat ring extensions R'/R of ramification index 1 (i.e. if one drops the condition of being finitely generated) (cf. prop. 3.6/3 and prop. 3.6/4).
- prove the "baby case" of the approximation theorem for excellent DVRs (cf. prop. 3.6/8, cor. 3.6/9 and cor. 3.6/10). Stress that the only non-trivial argument in these proofs is our Main Theorem!
- state theorem 3.6/12 and 3.6/16, which are reduced by induction (and rather technical arguments) to the "baby case". An overview over the main arguments would be appreciated.

For further informations, you may consider [Art] or (for a different approach to this theorem) [Rot].

References

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[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud, Néron Models, Ergebnisse der Mathematik und ihrer Grenzgebiete, series 3, vol. 21, Springer, Berlin, 1990

[Nér] A. Néron, Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Publ. Math. IHES, vol. 21, pp. 5-128 , 1964

[Rot] C. Rotthaus, On the approximation property of excellent rings, Invent. Math. 88, pp. 39-63, 1987