Kleine AG: Néron Models of Elliptic Curves

Organisation:
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Given an Abelian variety A over the generic point of a Dedekind scheme S, one would like to spread it out as an Abelian scheme to the whole of S. It turns out that this is not always possible (e. g., over $\mathbf{Z}$, there are no nontrivial Abelian schemes at all). But if one relaxes the conditions, i. e. one demands only that the group scheme is smooth, not necessarily proper, there is such a model $\mathcal{A}$, and it suffices a universal property, namely that $\mathrm{Hom}_K(X_K, A_K) = \mathrm{Hom}_S(X, \mathcal{A})$ for X/S smooth, called the **Néron mapping property**.

1. Arithmetic surfaces

All references refer to [Liu], Chapter 9.

- Explain 1.1–1.5 without proof: Define and motivate the intersection product on regular surfaces and state its properties.
- State 1.12 without proof: the definition of the intersection product with a vertical divisor.
- Prove Proposition 1.21: Calculate the intersection product of certain vertical divisors of an arithmetic surface.
- Prove Proposition 1.30 and Corollary 1.32: Calculate the intersection product of a horizontal divisor with the special fibre.
- Explain 3.1, 3.12–13: Introduce exceptional divisors and (relatively) minimal surfaces.
- State Prop. 3.19 and sketch its proof: The existence of a relatively minimal model.
- State Proposition 3.21 without proof: The existence and uniqueness of a minimal model. Also show Corollary 3.26, emphasising that the construction of 3.19 gives the minimal model.

2. The minimal regular model and Weierstraß models of an elliptic curve

All references refer to [Liu], Chapter 9.

- Define Weierstaß models of elliptic curves.
- Prove Proposition 4.26 and state Proposition 4.30 without proof: An alternative characterization of Weierstraß models.
- Follow p.446–447: Define the discriminant and the minimal Weierstraß model. Prove the existence of a minimal regular model.

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- Prove Theorem 4.35 and Corollary 4.37: Uniqueness and a condition for existence of the Weierstraß model as well as a comparison of the minimal Weierstraß model (if it exists) with the minimal regular model.

3. Existence of the Néron model of elliptic curves

All references refer to [Liu], Chapter 10.

- Prove Lemma 2.12: The open subscheme $\mathcal{N}$ of the smooth points of the minimal regular model carries a group scheme structure and $\mathcal{N}(S) \to E(K)$ is an isomorphism.
- Prove [BLR], 4.4, Theorem 1 on spreading out rational maps from smooth schemes to group schemes defined in codimension 1 and the generic points of all fibres.
- Deduce Theorem 2.14: The group scheme $\mathcal{N}$ from above is the Néron model.
- Explain the Summary.

4. Reduction of the minimal regular model

All references refer to [Liu], Chapter 10.

- Explain section 2.1: the dual graph of $\mathcal{E}_s$.
- Prove Lemma 2 on the reduction of the minimal Weierstraß model.
- State Definition 2.2 on the reduction type.
- Explain the rest of this section up to p. 487: classification of the possible dual graphs of $\mathcal{E}_s$, only briefly mentioning the Kodaira types with an "*" without proof. See also [Sil1], p. 448, Table 15.1.
- Explain Example 2.16.

5. The component group

All references refer to [Liu], Chapter 10; see also [Sil1], p. 448, Table 15.1.

- State without proof 2.17–2.21: existence and properties of the scheme of connected components.
- Prove and explain Remark 2.21–Lemma 2.25: classification of the possible component groups.
- Prove Proposition 2.26: the filtration of $E(K)$.

References

[BLR] Bosch S., Lütkebohmert W., Raynaud M.: *Néron models.*

[Liu] Liu Q.: *Algebraic Geometry and Arithmetic Curves*, 2nd edition.[3]

[Sil1] Silverman J.: *The Arithmetic of Elliptic Curves*, 2nd edition.[4]

[Sil2] Silverman J.: *Advanced Topics in the Arithmetic of Elliptic Curves.*[5]

[3] Note the errata `http://www.math.u-bordeaux1.fr/~qliu/Book/errata-third.pdf` and `http://www.math.u-bordeaux1.fr/~qliu/Book/errata-third-b1.pdf`

[4] Note the errata `http://www.math.brown.edu/~jhs/AEC/AECErrata.pdf`

[5] Note the errata `http://www.math.brown.edu/~jhs/ATAEC/ATAECErrata.pdf`