# Kleine AG: Motives

Organization:
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In this Kleine AG we want to study motives, i.e. (naively speaking) an attempt to create a universal cohomology theory. But as there are a lot of different definitions with different (dis)advantages, for example Chow motives, Tate motives, absolute Hodge motives, Voevodsky's geometric motives, Nori motives, ... we cannot get a comprehensive overview of the current state of art within just one day. Instead we try to answer questions like:

- What is the idea behind motives? For this we have a look at Grothendieck's original intentions and the definition of mixed Tate motives.
- What kind of theorems are the goal whens studying motives? Here we will only give two examples concerning the ring of periods, namely the conjectures of Grothendieck and Kontsevich-Zagier. Several others like the standard conjectures and their impact have to be left as a topic for further studies to the interested participant.
- What applications do motivic theorems have? We discuss some results on (in)dependence of zeta values. Compared to other applications this has the advantage that the ideas are reasonably accessible and show how to connect the notions above with other fields.

### 1. Grothendieck's philosophy of motives (and a brief introduction to derived categories) (45 minutes)

Explain Grothendieck's philosophy for motives, see [Bar, Introduction and chapter 1].
At the end give an overview over derived categories, because this will be needed in the next talk: Define triangulated categories [Huy, Def. 1.32] (omitting details when too technical), give the construction of a derived category of an abelian category [Huy, Thm 2.10] and explain a few properties like existence of a cohomology functor, e.g. [Huy, Cor 2.11].

### 2. Geometric motives and mixed Tate motives (45 minutes)

Define the (derived) category of geometric motives and mixed Tate motives. The constructions are explained in [HS, 6.2, 6.4], [Lev, 4.5, 5].
At least state that using so-called t-structures, one can define an actual abelian category of mixed Tate motives, when the ground field is a number field (or $\mathbb{Z}$). If time permits, give details what t-structures actually are and how one can reconstruct the original abelian category from its derived category. See [Lev, 5] and [DG, 1] for details (and note that the Beilinson-Soulé vanishing is known for mixed Tate motives over number fields).

### 3. The motivic Galois group (and a brief introduction to Tannakian formalism) (30 minutes)

Give an overview over the Tannakian formalism, i.e. explain what turns $\mathrm{Rep}(G)$ (for an algebraic group $G$) into a (neutral) Tannakian category (i.e. why it satisfies [DM, Def. 2.19]) and state the Tannaka reconstruction theorem [DM, Prop. 2.8 and Thm. 2.11]. Then define the motivic Galois group of a variety, cf. [Ayo, 3.2, 3.3] and [DG, 2] [3]. Details are given as well in [DM, 6.22-6.24], though be aware that they use absolute Hodge motives instead of mixed Tate motives. Explain the example of $\mathbb{P}^n$ over the ground field, i.e. decompose the motive of $\mathbb{P}^n$ into Lefschetz motives, which will generate the Tannakian category, that can then be identified with $\mathrm{Rep}(\mathbb{G}_m)$[4].

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[3] [DG] defines as well unipotent parts of motivic Galois groups, which you can safely ignore for this talk.
[4] For more details, contact the first organizer.

**4. The conjectures of Grothendieck and Kontsevich-Zagier (60 minutes)**

Each smooth variety over $\mathbb{C}$ has an associated ring of periods whose elements are obtained by integrating certain differential forms on classes in singular homology. Grothendieck's conjecture now states that the transcendence degree of the periods equals the dimension of the motivic Galois group.
First define the ring of periods for a smooth variety over $\mathbb{C}$, then reformulate this construction in motivic terms and state the Grothendieck conjecture. Finally relate the Grothendieck conjecture to the conjecture of Kontsevich-Zagier, which essentially states that all algebraic relations among the periods come from geometry. More precisely define the ring of abstract periods and state the conjecture of Kontsevich-Zagier. Then rephrase Grothendieck's conjecture in terms of the torsor of periods, relate it to the ring of abstract periods and finally explain why the Grothendieck conjecture is equivalent to the conjecture of Kontsevich-Zagier assuming that the ring of abstract periods is an integral domain. Refer to [Ayo, 1.1-1.3, 2.1, 3.5, 4] for details.

**5. Algebraic (in)dependance of multi-zeta values (45-60 minutes)**

Summarize Brown's article: The main (though very technical) part is getting the $\mathbb{Q}$-basis $\{\zeta^{\mathfrak{m}}(n_1,\ldots,n_r), n_i \in \{2,3\}\}$ for all motivic multi-zeta values, cf. [Bro, Thm 1.1]. Then note that there is a period morphism mapping motivic multi-zeta values to usual multi-zeta values, giving [Bro, Conj. 2].
To even go a step further, note that motivic multi-zeta values can be seen as the periods coming from a certain mixed Tate motive ${}_0\Pi_1$. Now observe that by the easy direction of Grothendieck's conjecture [Ayo, Remark 23], we obtain [Bro, Conj. 1]. This implies that all periods of mixed Tate motives over $\mathbb{Z}$ actually are multi-zeta values, i.e. a $\mathbb{Q}$-linear combination of $\{\zeta(n_1,\ldots,n_r), n_i \in \{2,3\}\}$, cf. [Bro, Cor 1.2].
The full conjecture of Kontsevich-Zagier implies even more: All algebraic relations between periods already exist on the level of motives, hence algebraic independence of motivic multi-zeta values implies algebraic independence of traditional multi-zeta values. Thus we obtain a conditional proof, that $\{\zeta(n_1,\ldots,n_r), n_i \in \{2,3\}\}$ are $\mathbb{Q}$-linearly independent.

# References


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