Kleine AG: mod p local Langlands correspondence over $\mathbb{Q}_p$

January 6, 2014

Organisation:
Alexander Ivanov[1]
Robert Wilms[2]

The goal of this Kleine AG is to give a first insight into the mod p-Langlands program. Let F be a local field with residue characteristic p and let k be an algebraically closed field of characteristic p. There is a conjectural correspondence between certain n-dimensional representations over k of the absolute Galois group G_F of F and certain representations of the group $\mathrm{GL}_n(F)$. The standing of the actual resarch is as follows. Almost everything is known about the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ and much less is known if $n > 2$ or $F \neq \mathbb{Q}_p$. We will only consider the "baby" case $\mathrm{GL}_2(\mathbb{Q}_p)$. In the first three talks the goal is to explain the modular representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$. In the fourth talk we will give an ad hoc "definition" of the Langlands correspondence and see Breuil's "programme", which explains an (a priori conjectural) way of a natural realization of this correspondence. Then in the last talk we will introduce (ϕ, Γ)-modules and see how they lead to such realization, and hence to a proof of Breuil's "programme".

For the first three talks the main reference is [He]. For the last two talks the main reference is [Br]. Each talk should take 45 minutes.

First talk. Motivation, Induction, Weights

After a brief motivation, the goal of this talk will be to recall necessary preliminaries, such as induced representations, Frobenius reciprocities and weights. A further goal is to classify weights of $\mathrm{GL}_2(\mathbb{F}_p)$ and to determine the weights contained in a principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The main reference for this talk is [He] Sections 1 - 4. Here is the precise content:

- Motivate the study of the p-adic and mod-p local Langlands programs by briefly recalling the (proven) ℓ-adic case ($\ell \neq p$) and pointing out the difference to the mod p case (section 1).

- Recall briefly the space of (compactly) induced functions and both Frobenius reciprocities without proof (section 3). There is no need to introduce the notation $[\gamma, x]$.

- Show that each mod-p representation of a pro-p-group has a fixed vector (Lemma 10).

[1] ivanov@ma.tum.de
[2] Robert-wilms@t-online.de

- Introduce weights (Definition 11) and show the extremely useful Corollary 12.
- Give the classification of weights for $\mathrm{GL}_2(\mathbb{F}_p)$ (Proposition 14).
- Determine the weights of principal series representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (Proposition 17).

Use the notations from the end of Section 2 and from Section 4. As the proofs are relatively simple, try to give as much arguments as time permits. In particular, it would be nice to see (the application of) the Mackey decomposition in the proof of Proposition 17.

Second talk. Irreducibility of Principal Series

The goal of the talk is to prove irreducibility of the principal series representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Therefore, one will need to introduce the Hecke algebra of a representation and study it via the Satake transform. The main reference for this talk is [He] Sections 5 - 8.

- Define the Hecke Algebra $(\mathscr{H}(V), *)$ associated with a representation V and sketch very briefly Proposition 18.
- State (after recalling the Cartan decomposition) the structure Theorem 21 and explain the strategy of its proof via the mod p Satake transform.
- Using Proposition 22 as a lemma, explain the relation between compact and parabolic inductions (Section 6). State Corollary 26.
- Define the Steinberg representation (Definition 27) and state without proof Theorem 28.
- Introduce the (bi-)module of intertwining operators $\mathscr{H}(V, V')$ and state Proposition 31.
- State Proposition 34, thereby weakening the assumptions of Corollary 26, and give as much details of the proof as possible.

Third talk. Classification Results

In this talk the theory developed in the last two ones culminates in a classification of all irreducible admissible mod-p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.There are two theorems for this, the classification theorem of Barthel-Livné dividing all (irreducible, admissible) representations in principal series, characters, twists of the Steinberg representation and the supersingular ones. This theorem should be proven. The second one is the theorem of Breuil, with a much more technical proof, which describes explicitely all of the supercuspidal representations. The reference for this talk is [He] Sections 9 and 10.

- Give the definition and the properties of admissibility.
- Define supersingular representations and prove theorem of Barthel-Livné classifying the mod-p admissible irreducible representations of $GL_2(\mathbb{Q}_p)$.
- Define supercuspidal representations and deduce that they are exactly the supersingular ones.

- State the theorem of Breuil, describing all supersingular representations. If time permits, it would be nice to see the main ideas behind its proof.
- Instead (or parallel to) giving the proof, it would also be interesting to compare these classification results to the representation theory of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{C}$ (cf. [BH] Chapter 2) or to the ℓ-adic case, which can be found in [BH] (in particular, see Classification theorem following 9.11 and Theorem 11.4).

Fourth talk. Langlands Correspondence I

The goal of this talk is to "define" the mod-p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and to state Breuils "programme", which should realize it. The main reference for this talk is [Br] §2.1 and §2.2.

- Define the Galois representations ρ_r (page 6, end of §2.1). [be aware of the slightly different notation. In particular, ω is introduced on p. 4]
- Recall the Langlands correspondence in dimension 1, i.e., the class field theory, via the reciprocity map (cf. p. 4, second paragraph).
- State the local Langlands correspondence in dimension 2 (Definition 2.2).

From now on and until the end of the next talk, our goal will be to give an overview of realizing the above correspondence via the theory of (ϕ, Γ)-modules. Therefore, we will have to change the setting and consider now $\mathcal{O}_E$- resp. E-valued representaions (E a sufficiently big extension of $\mathbb{Q}_p$, cf. p. 4). Also from now on we give up concrete details and aim only an overview.

- Explain §2.2. Try to omit techincal details (such as exact definitions of algebraic and analytic vectors[3]). Concentrate on the last part of §2.2, i.e., Breuil's "programme" which should associate to a 2-dimensional Galois representation V (satisfying some conditions) an admissible unitary Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, which satisfy the conditions (i)-(iv).

For us (as we are interested in the mod-p case) the most interesting condition is (iii). Thus the association $V \mapsto B(V)$ would realize the correspondence defined above. In the next talk we will sketch the construction $V \mapsto B(V)$ via the étale (ϕ, Γ)-module associated to V.

Fifth talk. Langlands Correspondence II

The main reference for this talk is [Br] §2.3.

- Define (étale) (ϕ, Γ)-modules.
- State the equivalence of categories of integral Galois representations and (ϕ, Γ)-modules (Theorem 2.4, without proof).

[3] But observe that 'locally algebaic' has essentially the same meaning as in the classical case the K-finite vectors.

- Define the E-vector spaces $D(V) \boxtimes_\chi \mathbb{P}^1 \supseteq D^\sharp \boxtimes_\chi \mathbb{P}^1$ and $B(V)$. Left out as much technical details as possible.
- State Colmez's theorem 2.5, which shows that the Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ satisfies the requirements of Breuil's "programme" (without proof).

References

[Br] Breuil Ch.: *The emerging p-adic Langlands programme*, Proc. of the ICM, Hyderabad, India, 2010, http://www.math.u-psud.fr/ breuil/PUBLICATIONS/ICM2010.pdf.

[BH] Bushnell C.J. and Henniart G.: *Local Langlands conjecture for GL_2*, Springer, 2006.

[He] Herzig F.: *Mini-course: p-modular representations of p-adic groups*, http://www.math.toronto.edu/ herzig/singapore-notes.pdf