Kleine AG: Log structures on schemes

Organization:
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In this Kleine AG we want to study logarithmic structures on schemes, i.e. schemes X together with a sheaf of monoids $\mathcal{M}_X$ and a morphism $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ identifying $\mathcal{O}_X^*$ with its preimage under α. This has one very important consequence: Mildly singular schemes behave just like smooth schemes in the logarithmic setting!
This has some very interesting applications:

- If you have a class of smooth varieties, then their moduli space is usually not proper. However allowing your varieties to be only log smooth often gives you a nice compactification.
- Assume now that your moduli space of smooth objects is smooth itself. But even then its compactification usually contains singularities at the boundary. However we can still hope the compactification to be log smooth.
- Adding logarithmic structures allows us to extend de Rham cohomology to mildly singular schemes.

A note on the literature: We will mainly follow [AO10], because it is a very concise summary of the most important facts. To get more details and proofs of the general theory, have a look at [Ka88] and Ogus' unfinished lecture notes [Og06].

1. Introduction to logarithmic structures (45 minutes)

Define pre-logarithmic structures, logarithmic structures, log schemes [AO10, 2.2] and log structures induced by pre-log structures. Then introduce morphisms of log schemes [AO10, 2.9], charts [AO10, 2.12] and fine (and saturated) log schemes [AO10, 2.16]. Give as many examples as you can for all these notions.

2. Log smooth morphisms (45 minutes)

Introduce log derivations on log schemes [AO10, 3.3]. Then construct the sheaf of log differentials $\Omega^1_{X/Y}$ [AO10, 3.4]. There are a lot of different instructive examples with $Y = \operatorname{Spec} k$, e.g. the singular variety in [AO10, 3.7][3] or $(\mathbb{A}^1_k, \mathbb{N} \to k[\mathbb{N}] \cong k[t])$ which contains logarithmic differentials.
Then define log smooth morphisms [AO10, 3.11-3.12] and stress that for morphisms of the form $f : \operatorname{Spec}(P \to R[P]) \to \operatorname{Spec}(Q \to R[Q])$ we can translate log smoothness into a purely monoid-theoretic condition. Explain how the standard example of a normal crossing singularity can be turned into a log smooth variety [AO10, 3.7], cf. as well [AO10, 2.7] resp. [Og06, IV.3.1.20]. More examples of log-smooth morphisms can be found in [Og06, IV.3.1.15-IV.3.1.19]. State that $\Omega^1_{X/Y}$ is locally free if X is log smooth over Y, [Ka88, 3.10] resp. [Og06, IV.3.2.1].

3. Properties of log smooth curves (45 minutes)

Define integral and flat morphisms [AO10, 3.15-3.18]. Then introduce log curves [AO10, 4.5], which will automatically have no singularity worse than a nodal one [AO10, 4.7]. State the comparison between the sheaf of log differentials $\Omega^1_{X/k}$ and the usual dualizing sheaf [AO10, 4.9]. Explain the main ideas of the proofs of these statements, cf. [Ka99, 1.3, 1.13]. Finally sketch the construction of a canonical log structure on an arbitrary given curve with n marked points, cf. [AO10, after 4.10] or for more details [Ka99, §2].

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[3] Note that there is not much difference between the examples considered in [AO10, 2.7] and [AO10, 3.7], as the generating differentials coincide. Passing to the normal crossing divisor just adds one relation.

4. Moduli of log smooth curves and the Deligne-Mumford compactification (45 minutes)

The main goal of this talk is to prove that the moduli space of log smooth curves coincides with the Deligne-Mumford compactification of the moduli space of smooth curves, cf [AO10, 4.13] for a more precise statement.
To see the existence of a moduli space of log smooth curves, give a proof of [AO10, 4.11] resp. [Ka99, 2.3, 2.6]. Deduce that there exists a stack $\mathcal{M}_{g,n}^{bas}$ representing basic stable log curves (of type (g, n)) [Ka99, 4.2]. Next recall the main properties of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of n-marked smooth curves of genus g, cf. [ACG11, §XII]. The main ingredients are

- the definition of a stable curve with n marked points, providing the category that has the Deligne-Mumford compactification as a moduli space,
- the stable reduction theorem, providing properness of $\overline{\mathcal{M}}_{g,n}$ and
- the covering by $H_{g,n}$, providing the description of the boundary as a normal crossing divisor[4].

Finally conclude that there is an isomorphism between $\mathcal{M}_{g,n}^{bas}$ and $\overline{\mathcal{M}}_{g,n}$ as explained in [Ka99, 4.5]. Note that it follows from talk 2, that $\mathcal{M}_{g,n}^{bas}$ is log smooth[5].

5. Analytifications and log de Rham cohomology (45-60 minutes)

This talk is meant to give an overview over two other interesting constructions: The space X^{log} (often called the Kato-Nakayama space, cf. [KN99]) associated to a log scheme over $\mathbb{C}$ and the log de Rham cohomology. To get a rough first idea, you can look at the section "Log de Rham complex" in [AO10, §9]. Start by recalling the definition of the analytic topology on $X^{an} = X(\mathbb{C})$ [Og06, intro. to II.3.2]. Mimic this when defining $X^{log} = X(\mathbb{C}^{log})$ and its topology [Og06, II.3.1.1, II.3.2.3], [KN99, §1] (which is in general not a complex analytic manifold, even though it comes with a sheaf of functions on it). Explain the relationship between X^{an} and X^{log}, so that the audience gets a picture of X^{log}. For example you may talk about...

- an example like $X = \mathrm{Spec}\,(\mathbb{N} \to \mathbb{C}[\mathbb{N}])$ with $X^{log} = \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \xrightarrow{mult} X^{an} = \mathbb{C}$.
- the high-level point of view, as in the explanation directly before [Og06, II.3.4.2] resp. [KN99, 1.2].
- some nice results regarding the morphism $X^{log} \to X^{an}$ and its fibers, cf. [Og06, II.3.2.5, II.3.2.6][6] and [KN99, 1.2.3].
- an alternative construction of X^{log} via charts, as given in [Og06, II.3.2.6(2.)].

In the second part of the talk, discuss the log de Rham cohomology. Begin with the construction of the log de Rham complex [Og06, V.1.1] (explaining the definition of the differential, but omitting technical details). Then explain theorem [Og06, V.0.2.1]. Ogus gives a rough sketch of the proof, but unfortunately most details are missing. We suggest focusing on the arrow comparing log de Rham with analytic de Rham cohomology and either trying to understand Ogus' sketches or follow the original proof [KN99, 4.7-4.9].

References

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[4] Alternatively the description of the boundary can be deduced from properties of Kuranishi-families.

[5] Actually one can prove that $\overline{\mathcal{M}}_{g,n}$ itself is smooth in the usual sense.

[6] see [Og06, II.3.1.2(2.)] for the definition of $\mathbf{S}_{X,x}$.

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