KLEINE AG: K-GROUPS OF RINGS OF INTEGERS ARE FINITELY GENERATED

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This *Kleine AG* is concerned with Quillen's monumental proof in [Qui73] that the K-groups of rings of integers (of number fields) are finitely generated. As K-groups are notoriously difficult to compute (to this date, the K-theory of $\mathbb{Z}$ is the object of important conjectures), this was actually a big thing.

We propose the following five talks of 60 minutes each.

1. Homotopy theory and K-theory

This talk should start by explaining Quillen's "Theorem A" and "Theorem B" (cf. e. g. [Sri08, chapter 6]) and the notion of homotopy colimits (use something like [MP12, chapter 2] or [Dug08]) — we don't need to understand technical details or proofs here, but more a rough idea of the concepts involved. Some examples from topology might prove enlightening. Furthermore, K-theory should be introduced via Quillen's "Q-construction" (cf. e. g. [Wei13, IV.6] or [Sri08, section 4]). Explain why $K_0(\mathcal{O}_K) \cong \mathbb{Z} \oplus \mathrm{Cl}_K$ following Weibel. State results about other low K-groups and mention the Kummer-Vandiver and Lichtenbaum conjectures (or any other results the speaker finds appropriate to motivate algebraic K-theory for that matter).

2. The Solomon-Tits Theorem

The aim of this talk is to prove that if V is an n-dimensional vector space, then its attached building $\boxed{V}$ (of proper subspaces) has the homotopy type of a bouquet of $n-2$-spheres. Its reduced homology is hence concentrated in degree $n-2$, and we call $st(V) = \tilde{H}_{n-2}(\boxed{V})$ the *Steinberg module* attached to V. It carries a natural GL_n-action. A different interpretation of the Steinberg module can be given as $st(V) = \tilde{H}_{n-1}(BJ(V))$, where $J(V)$ is the category of pairs of subspaces of V, one of which is non-trivial, with a suitable partial order. This is all laid out quite nicely in [Qui73, section 2].

3. The Borel-Serre Compactification

To study the cohomology of arithmetic groups, Borel and Serre introduced the Borel-Serre compactification $\overline{X}$ in [BS73] of a certain homogenous space X under an algebraic group G. Quillen's proof relies crucially on a duality result for arithmetic groups and the explicit determination of the dualizing module, which in essence is combinatoric in nature.

The center piece of this talk is to understand the following chain of isomorphisms (using Poincaré duality without much mention), which will eventually determine the dualizing module:

$$H_c^{d-n}(\overline{X}, \mathbb{Z}) \cong \tilde{H}_{n-1}(\partial\overline{X}, \mathbb{Z}) \cong \tilde{H}_{n-1}(|T(G)|, \mathbb{Z}) \cong \tilde{H}_{n-1}(B\mathfrak{P}(G), \mathbb{Z}) \cong st(V).$$

The last isomorphism is explained in the proof of theorem 1 in [Qui73].

The talk should first focus on the construction of $\overline{X}$ as done in [BS73, chapter 2]. While it may be necessary to recap a few basic notions concerning arithmetic groups, we will only need the case of GL_n, so please illuminate the talk with more concrete interpretations as done in [Sap03, section 6]. Afterwards explain [BS73, theorem 8.4.1 and corollary 8.4.2] in as much detail as time allows.

4. Duality for Arithmetic Groups

First sketch the abstract duality results of Bieri and Eckmann (cf. [BE73]) as used in the context of arithmetic groups ([BS73, theorems 11.4.1 and 11.4.2]). Sketch why an arithmetic group G contains a normal torsion-free subgroup of finite index Γ (cf. e. g. [Sou07, theorem 9]). It is of crucial importance that $\overline{X}/\Gamma$ be compact, so that $\overline{X}$ becomes a universal covering of the $K(\Gamma,1)$-space $\overline{X}/\Gamma$. Therefore explain [BS73, theorem 9.3] as well. Show that with $I = H^{d-n}(\Gamma,\mathbb{Z}[\Gamma]) \cong H_c^{d-n}(\overline{X},\mathbb{Z})$ (which is just $st(V)$ by the previous talk) we have

$$H_i(\Gamma,I) \cong H^{d-n-i}(\Gamma,\mathbb{Z}),$$

which is finitely generated as $\overline{X}/\Gamma = B\Gamma$ is compact.

5. The Long Exact Sequence and Proof of the Main Theorem

This talk first concerns itself with the long exact sequence in [Qui73, theorem 3]. Absolutely focus on number fields and barely mention that the result also holds for division algebras. The talk should be at least as vague as Quillen on the spectral sequence — there really is no need to dive deeper into homotopy theory. We don't want to see the homological arguments in agonizing detail — it will be much more interesting to show how all the previous pieces fit together to prove the main theorem. Follow [Qui73, proof of theorem 1 from theorem 3] for that — but ignore references to Raghunathan, as finite generation of the cohomology groups was subject of the previous talk.

References

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