Kleine AG: Serre's Modularity Conjecture

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Synopsis

The goal of this Kleine AG is to understand the statement and some of the consequences of Serre's Modularity Conjecture, following Serre's original paper [Ser87]. Among the consequences is a proof of Fermat's Last Theorem.

Throughout, fix a prime number p. We begin by stating an imprecise version of the conjecure:

Conjecture 1. *Every continuous, odd, irreducible Galois representation*

$$\rho\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{GL}_2(\overline{\mathbb{F}}_p)$$

is modular.

Here we call ρ *modular* if there exists a $\operatorname{mod} p$ cuspidal Hecke eigenform f whose associated Galois representation ρ_f is isomorphic to ρ.

Roughly speaking, a $\operatorname{mod} p$ modular form f is the reduction $\operatorname{mod} p$ of a modular form F in the usual sense. Such a modular form F has a level N, a weight k and a character ε. We call the tuple (N, k, ε) the *type* of F.

But given ρ, how can we ever hope to find the modular form f among the space of all modular forms? To restrict our search for f, it is essential to have a more precise version of the conjecture, prescribing the type (N, k, ε) of the sought $\operatorname{mod} p$ modular form f. To do this, Serre writes down a recipe for obtaining integers $N(\rho)$ and $k(\rho)$ and a character $\varepsilon(\rho)$ associated to ρ. Then the precise form of the conjecture is:

Conjecture 2. *Let*

$$\rho\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, odd, irreducible Galois representation. Then there exists a $\operatorname{mod} p$ *cuspidal Hecke eigenform f of level $N(\rho)$, weight $k(\rho)$ and character $\varepsilon(\rho)$ such that*

$$\rho_f \cong \rho.$$

From conjecture 2 it is possible to deduce both Fermat's Last Theorem and the Shimura-Taniyama-Weil conjecture.

Both conjecture 1 and 2 are now known to be true. The proof involves the work of several people and a lot of intermediate results, for example the equivalence of conjecture 1 and 2. The proof was completed by Khare-Wintenberger, see *Invent. Math., 178(3), 2009.*

In this Kleine AG we will not talk about the proof of the conjecture, but rather want to understand the statement of the conjecture together with the necessary background material and applications.

Talks

The main reference is Serre's article [Ser87] (in French). An English translation is available by Ghitza [Ghi12]. However, maybe the most accessible is the essay by Best [Bes15].
As a guideline, every talk should last between 60 and 75 min.

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Talk 1: Galois representations. Following [Bes15, §2.2], define mod p Galois representations; explain the words continuous, odd and twists. Recall the definition of the higher ramification groups [Bes15, 2.2.5] and define $N(\rho)$ (and the Artin conductor) as in [Bes15, §4.4], see also [Ser87, §1.2]; define when ρ is unramified at ℓ and explain the unramified case [Bes15, 4.4.1]. Define $\varepsilon(\rho)$ and $h(\rho)$ as in [Bes15, §4.5], see also [Ser87, §1.3]. Finally, explain wild/tame inertia groups and fundamental characters of the tame inertia, see [Bes15, 2.2.8-2.2.16], [Ser87, beginning of §2.1], [Ser72, §1.1-1.3,1.7-1.8].

Main references: [Bes15, §2.2 until 2.2.16; §4.4 and §4.5]; [Ser87, §1]; [Ser72, §1.1-1.3,1.7-1.8].

Talk 2: mod p modular forms and Serre's conjecture (imprecise version). Following [Bes15, §2.1], define mod p modular forms as the reduction of complex modular forms; define the minimal weight (= filtration in [Bes15, (2.1.6)]); recall Hecke operators and eigenforms. State Deligne's result on the existence of a Galois representation ρ_f attached to an eigenform f [Bes15, 3.2.1], see also [Ser87, (3.1.7)]; then state Serre's conjecture in the imprecise form. Finally, sketch a few of the properties of ρ_f, namely [Bes15, 3.2.2] and the results of [Bes15, §4.3][3], explaining how one can try to read off the type (N, k, ε) of the modular form f from its Galois representation ρ_f.

Main references: [Bes15, §2.1, §3, §4.3]; [Ser87, §3.1].

Talk 3: Weight of a representation and Serre's conjecture (precise version). This is probably the most technical talk. Start by recalling tame inertia and fundamental characters from talk 1, references see above. Explain why the wild inertia acts trivial on semisimple representations, see [Ser72, prop. 4] and [Bes15, 4.6.1]; deduce [Ser87, Prop. 1]=[Bes15, 4.6.2], then start to explain the recipes for $k(\rho)$, see [Bes15, §4.6.1-4.6.3] and [Ser87, §2.2-2.4]. For the last case ([Bes15, §4.6.3], [Ser87, §2.4]), recall Kummer theory [Bes15, 2.2.6] and explain peu/très ramifié [Bes15, p. 12]. Finally, state and prove the results of [Ser87, §2.5], and end by stating Serre's conjecture in its precise form.

Main references: [Bes15, §2.2 from 2.2.6; §4.6]; [Ser87, §2.1-2.5].

Talk 4: Representation associated to an elliptic curve and Fermat. Start by explaining the Galois representation associated to an elliptic curve [Ser87, §2.9]. The first goal of this talk is an (at least partial) proof of [Ser87, prop. 5], calculating the weight of this representation. This can be done in two ways: Either following [Ser72, §1.9, 1.11, 1.12], explicitly study this representation and then follow the recipe of the previous talk. Or following [Ser87], first prove prop. 4 using *finite group schemes of type (p,p)*, a term defined in [Ray74] and then deduce prop. 5. Both approaches also make use of the Tate model of an elliptic curve, which can be found in [Sil94, §V], in particular theorems V.3.1(d) and V.5.3.

Then explain how to get an elliptic curve E from $A, B, C \in \mathbb{Z}$ with $A+B+C=0$ as in [Ser87, §4.1] and calculate the corresponding reduction properties; prove Prop. 6 and calculate the type $(N(\rho), k(\rho), \varepsilon(\rho))$ of the associated rep. [Ser87, (4.1.10-12)]; state and prove [Ser87, thm 1] (Fermat). Time permitting, state and sketch a proof of [Ser87, thm 4] (Shimura-Taniyama-Weil conjecture).

Main references: [Ser87, §2.8, 2.9, 4.1, 4.2, 4.6]; [Ser72, §1.9-1.12].

References

[Bes15] Alex J. Best. Serre's conjecture. https://alexjbest.github.io/partiiiessay/essay.pdf, 2015.

[Ghi12] Alex Ghitza. On the two-dimensional modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ (translation). https://aghitza.github.io/pdf/other/serre.pdf, 2012.

[Ray74] Michel Raynaud. Schémas en groupes de type $(p, \dots, p)$. *Bull. Soc. Math. France*, 102:241–280, 1974.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[Ser87] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.

[Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[3] Some of the words (Θ-operator and très ramifié) are not yet defined. If time permits, it would be nice to define Θ [Bes15, 2.1.7-2.1.9], but it is not essential. "Très ramifié" will be defined in the subsequent talk.