# Kleine AG: Fun with $\mathbb{F}_1$

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The idea of studying $\mathbb{F}_1$-geometry was originally suggested in 1956 by Jacque Tits on the basis of an analogy between symmetries in projective geometry and the combinatorics of simplicial complexes. More precisely, Tits observed the analogy between the symmetric group $S_n$ and the finite Chevalley group $\mathrm{GL}_n(\mathbb{F}_q)$, which suggests that one should consider $S_n$ as "$\mathrm{GL}_n$ over the field with one element". Another motivation comes from algebraic number theory and Arakelov geometry: since many years one knows that there are several strong analogies between the case of number fields and of function fields in one variable over finite fields. Nevertheless, there are also differences: in particular, in the function field case there is a finite base field $\mathbb{F}_q$ over which each object lies. This is extremly useful. In particular, this allows to define the Frobenius over $\mathbb{F}_q$, which acts on each object and provides us with a lot of additional structures, which allow to study the occuring objects. As an example of the power of these additional structures, one should mention the proof of the Weil conjectures, including the analog of the Riemann hypothesis for curves over finite fields, which were proven using those structures. On the other side, in the number field case there is no such base object, $\mathrm{Spec}\,\mathbb{Z}$ itself being the initial object in the category of rings. Now the hope is that one can enlarge the category of rings (resp. schemes) in an appropriate way such that there is a certain object "$\mathbb{F}_1$" in this category such that $\mathrm{Spec}\,\mathbb{Z}$ lies over $\mathrm{Spec}\,\mathbb{F}_1$ in a similar way as a curve over a finite field lies over $\mathrm{Spec}\,\mathbb{F}_q$.

Up to now, there are several approaches towards $\mathbb{F}_1$-geometry as explained in [LL]. In this Kleine AG we will study the approach given by Durov in his Ph.D. thesis [Du]. His motivation was in particular to give a new approach to Arakelov Geometry. For this purpose, he introduces the notions of generalized rings and generalized schemes (this is exactly the enlarging of the category of schemes in the abovementioned sense). To this extent, Durov considers the category of *monads* on a given category $\mathscr{C}$, which are simply endofunctors on $\mathscr{C}$ endowed with a multiplicative structure and a unit. Posing certain finiteness conditions, one obtains the subcategory of the *algebraic monads*. Restricting the category further, one gets the category of *additive monads*, which turns out to be equivalent to the category of semirings. In particular, any (commutative) ring gives rise to a (commutative) additive monad on the cagegory of sets. This motivates the

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definition of *generalized rings* as commutative algebraic monads. In particular, in this category one encounters many interesting (not necessarily new) objects as

- $\mathbb{N}$ – the natural numbers,

- $\mathbb{F}_\varnothing$ – "the field without elements" (initial object in the category of monads over sets),

- $\mathbb{Z}_\infty$ – "the completion at $\infty$ of the compactification of $\mathrm{Spec}\,\mathbb{Z}$" (an analog of $\mathbb{Z}_p$),

- $\mathbb{F}_1 := \mathbb{N} \cap \mathbb{Z}_\infty$ – "the field with one element",

- $\mathbb{F}_{1^n}$ – "finite extensions of $\mathbb{F}_1$",

- $\mathbb{F}_{1^\infty} := \varinjlim \mathbb{F}_{1^n}$ – "the algebraic closure of $\mathbb{F}_1$",

etc. As for rings, one can also consider the spectrum of generalized rings. This allows to define affine generalized schemes. Further fundamental constructions from basic algebraic geometry can be carried over to the category of generalized rings. This allows to construct the category of generalized schemes, in which in particular the Arakelov compactification $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ of $\mathrm{Spec}\,\mathbb{Z}$ lives.

Towards the end of this introduction, let us describe a problem of this approach. In the case of curves over finite fields, the fiber product $C \times_{\mathbb{F}_q} C$, which is used in the proof of the Weil conjectures, is a two-dimensional object over $\mathbb{F}_q$, in particular it is not just the curve $C$ itself. In the category of rings one trivially has $\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z} = \mathrm{Spec}\,\mathbb{Z}$, because $\mathrm{Spec}\,\mathbb{Z}$ is the final object. Thus the hope for a good $\mathbb{F}_1$-theory should include the fact that the fiber product $\mathrm{Spec}\,\mathbb{Z} \times_{\mathbb{F}_1} \mathrm{Spec}\,\mathbb{Z}$ in the appropriate "category over $\mathbb{F}_1$" is not just $\mathrm{Spec}\,\mathbb{Z}$ and that one has a non-trivial diagonal embedding of $\mathrm{Spec}\,\mathbb{Z}$ into this product. Unfortunately, in the category of Durovs enlarged rings, one still has $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}$.

The main reference will be the survey article [Fr] on the thesis mentioned above. [Du] can also be used for full details.

## First talk. Motivation (30 minutes)

The aim of this talk is to give a motivation of studying $\mathbb{F}_1$-geometry from Arakelov geometry as in [Du, Chapter 1]. Beginning with the idea of Arakelov geometry as in Section 1.1, one should explain how the results for a smooth projective curve as in Sections 1.2-3 could be transferred to the compactification of $\mathrm{Spec}\,\mathbb{Z}$ if this should exist as in Sections 1.4-5. Explain Section 1.6 if time permits.

## Second talk. Monads (60 minutes)

Begin with the definitions of monads and its modules as well as their subobjects and morphisms with examples as the beginning of [Fr, §3]. If this should help in understanding, one might refer to [Du, §3.1,3.3], where a monad is an algebra in the tensor category of endofunctors over a fixed category. The monad of words (see [Fr, Exemple 2], or more generally, [Du, 3.4.6]) will be used in the next talk so that this should also be explained. Then explain how the category of rings can be embedded in that of (algebraic) monads [Fr, §3.1]. Afterwards, define an algebraic monad and explain the equivalence between the category of algebraic monads and $\mathrm{Hom}_{\mathrm{Cat}}(\underline{\mathbb{N}}, \mathrm{Sets})$ as well as its consequences, i.e., the description of a monad as a collection of sets $\{\Sigma(n)\}_{n \in \mathbb{N}}$ with

a family of morphisms satisfying certain conditions [Fr, §3.2]. The proofs can be given if time permits.

## Third talk. Presentation of algebraic monads and generalized rings (45 mins)

The first part of the talk should be shortly devoted to a presentation of algebraic monads as well as related notions such as "finitely generated" and "finitely presented" [Fr, §3.3]. Then one should come to motivate and define generalized rings. For this, define the pseudoaddition of a monad as well as its hypo-, hyper- and additivity. Then prove that the additive monads correspond to semirings and those with a symmetry to rings [Fr, §3.4]. Before coming to the definition of generalized rings, give the definition of commutativity as well as some examples, then define a gereralized rings and fields [Fr, §3.5]. The last example to be given in this talk is $\mathbb{N}$ as a generalized ring [Fr, §4.1].

## Fourth talk. $\mathbb{Z}_\infty$, lattices and $\mathbb{F}_1$ (45 minutes)

The aim of this talk is to provide several examples of generalized rings inclusive $\mathbb{F}_1$. The first thing to be discussed in this talk is $\mathbb{Z}_\infty$ [Fr, §4.2]. As special cases of $\mathbb{Z}_\infty$-modules, one has the notion of $\mathbb{Z}_\infty$-lattices, for which one can also refer to [Du, Ch.2]. This example also leads to the new definition of a valuation ring [Fr, §4.3], which contains valuation rings in the non-archimedean case, but also works for the archimedean case, as well as $\mathbb{F}_\infty$ (loc.cit.). Then we come to $\mathbb{F}_1$ and its properties [Fr, §4.4], then its cyclotomic extensions [Fr, §4.5], and a finite presentation of $\mathbb{Z}$ over $\mathbb{F}_1$ [Fr, §4.6]. Several outlooks due to $\mathbb{F}_1$ can also be given if time permits. Finally, as a preparation for the construction of the compactification of $\operatorname{Spec}\mathbb{Z}$, introduce and discuss the generalized rings $A_N$ and $B_N$ [Fr, §4.7].

## Fifth talk. Generalized schemes (45 minutes)

The aim of this talk is to give the definition of generalized schemes and give as an example a construction of the compactification of $\operatorname{Spec}\mathbb{Z}$. For the notion of generalized schemes, one need the notions of a localization of a generalized ring, prime spectrum and generalized ringed space as explained in [Fr, §5.1]. Then two constructions of the compactification of $\operatorname{Spec}\mathbb{Z}$ should be sketched, namely the construction by hand [Fr, §5.2] and the construction obtained by the projective spectrum of a graduated generalized ring [Fr, §5.3]. The speaker should not get lost in technical details. The goal is to have a good overview of the mentioned sections at the end of the talk. If time permits, one may also give an overview of Section 5.4 in [Fr].

# References

[Du] N. Durov: *New Approach to Arakelov geometry.* arXiv:0704.2030v1 [math.AG], 2007.

[Fr] J. Fresán: *Compactification projective de* $\operatorname{Spec}\mathbb{Z}$ *(d'après Durov).* arXiv:0908.4059v1 [math.NT], 2009.

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