Bayerische Kleine AG

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Abelian Schemes over $\operatorname{Spec}\mathbf{Z}$

Organization:
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The class of elliptic curves provides a key example of Abelian varieties of dimension one whose rational points are well understood by Mordell-Weil Theorem. In the question of finding an integral model, there is however a famous Theorem of Tate [Tat74] that *there is no elliptic curve over* $\mathbf{Q}$ *with everywhere good reduction*. The aim of this *Bayerische Kleine AG* is to present a generalization of this result to higher Abelian schemes which has been independently given by Abrashkin and Fontaine [Abr87, Fon85]: *there is no non-trivial Abelian scheme over* $\operatorname{Spec}\mathbf{Z}$ *and other "small" number rings—i. e.* $\operatorname{Spec}\mathbf{Z}[i]$, $\operatorname{Spec}\mathbf{Z}[\sqrt{-3}]$, $\operatorname{Spec}\mathbf{Z}[\sqrt{-5}]$ *or* $\operatorname{Spec}\mathbf{Z}[\zeta_n]$ *for* $n \leq 7$.

This seminar presents in action the general theories of elliptic curves and of finite flat group schemes over a ring with a touch of ramification theory, in a way that it should benefit to every participant. We first follow the proof of Tate's Theorem via prime reduction and torsion points. We then recover these ideas in Fontaine's general approach for proper smooth group schemes and their finite flat group schemes of torsion points, whose rational points are controlled by higher ramification groups and Cartier duality: Fontaine's Theorem then follows from a contradiction on the number of rational points of extensions of 2-power order group schemes in an adequate category.

Our main reference is the lecture notes of René Schoof [Sch] on the proof of Fontaine and we restrict our study to $\operatorname{Spec}\mathbf{Z}$. Speakers will favour examples and should feel free to omit details of the proofs in case of time constrains.

Talk 1 – Introduction: Abelian schemes and elliptic curves. (60 minutes)
Define an elliptic curve as an Abelian variety of dimension one ([Mil86], § 1 and § 20) and explain that it can be represented as a smooth plane cubic by Weierstrass equations ([Sil09], § III.1). Present the notion of bad reduction in the example of elliptic curves ([Sil09], § III.2, "Singular Weierstrass Equations"), then explain the correspondence between divisors of the discriminant and primes of bad reduction ([Sil09], § III.1 or [Hus04] 5.§ 3). Give a sketch of Tate's Theorem by showing that there is a $\mathbf{Q}$-rational 2-torsion point and by using the Minkowski bound ([Sch], p. 2–5). Explain the equivalence with Fontaine's Theorem for Abelian schemes of dimension one.

Talk 2 – Finite flat group schemes and Deligne Theorem. (60 minutes)
Recall the definition of group schemes and Hopf algebras with examples ([Sch] § 2, and $G_{a,b}$ on p. 10), then Cartier duals ([Sch], p. 16) with the example $(\mathbf{Z}/m)^\vee = \mu_m$ ($\mathbf{Z}/m$ is a constant group scheme and μ_m is a diagonisable group scheme). Prove that for $m \in \mathbf{N}$, the m-torsion points $\mathscr{A}[m]$ of an Abelian scheme $\mathscr{A}/S$ is a finite flat group scheme over S, which is étale if and only if m is not divisible by the residue characteristics of S ([Mil86], Proposition 20.7). State Deligne's Theorem that a finite flat commutative group scheme of rank m is annihilated by multiplication by m ([Sch], p. 19–21). Explain

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the corollary on the annihilation of the differential module $\Omega^1_{A/R}$ of a Hopf algebra A over a ring R ([Sch], p. 25). Deduce that a finite flat group scheme over a field of characteristic 0 is étale. Explain the classification of finite flat group schemes of order 2 over a principal ideal domain: $G \cong G_{a,b}$ with $ab = 2$. For finite flat group schemes, see also [CSS97], *Finite Flat Group Schemes*, p. 121–154.

Talk 3 – Ramification theory and finite flat group schemes. (60 minutes)
Let Γ be a finite flat group scheme over a number ring $\mathfrak{O}_K$ and let $L = K(\Gamma(\bar{K}))$ and $G = \mathrm{Gal}(L/K)$. We study the ramification properties of the extension L/K. Recall briefly the principle of ramification theory ([Neu99], II §7–§ 10) and define the higher ramification groups $G^{(i)}$ and $G_{(i)}$ (note that Neukirch uses a different normalisation of the valuation than Schoof!), then $u_{L/K}$ for local fields, present the example of $\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p$ ([Sch], p. 34–35 or [Neu99], *loc. cit.*). State Fontaine's Theorem on the higher ramification groups for finite flat group scheme over $\mathbf{Z}_p$ killed by p^n, illustrate the Katz-Mazur example G_ε and state the two Corollaries on the bound for the valuation of the discriminant ([Sch], p. 36). State Fontaine's Proposition ([Sch], p. 37) and the triviality of $G^{(u)}$ for a finite flat commutative group scheme killed by $[p^n]$ ([Sch], Corollary p. 39).

Talk 4 – Application to 2-power simple finite flat group schemes. (60 minutes)
Let $\mathscr{A}$ be an Abelian scheme over $\mathrm{Spec}\,\mathbf{Z}$. Follow Fontaine's approach (1)–(6) ([Sch], p. 41) and show that ramification theory (Talk 3) implies that a finite flat simple group scheme over $\mathbf{Z}$ of 2-power order has order 2 and identifies either to $G \cong \mathbf{Z}/2$ or μ_2 ([Sch], (6) p. 41 and Theorem p. 43): denoting $\Gamma := \mathscr{A}[2^m]$ and $L := K(\Gamma(\bar{K}))$, show that L/K is unramified outside $p = 2$ (1), then "moderately" ramified over $p = 2$ (2), that the discriminant $|\Delta_{L/\mathbf{Q}}|^{1/[L:\mathbf{Q}]}$ is bounded (3), hence that $[L : \mathbf{Q}]$ is bounded by 4 and $\mathrm{Gal}(L/\mathbf{Q})$ is a 2-group (4). Mentioning (5), prove (6) and finally the Theorem on p. 43 which identifies a finite flat group scheme $G/\mathbf{Z}$ of order 2 with $G_{a,b} \cong \mathbf{Z}/2, \mu_2$.

Talk 5 – Fontaine's Theorem and extensions of 2-power finite flat group schemes over Spec Z. (60 minutes)
Explain the equivalence of categories between finite flat R-group schemes over a Noetherian ring and triples consisting of finite flat group schemes over $R[\frac{1}{p}]$ and $\hat{R} = \varprojlim_n R/(p^n)$ for $p \in R$ ([Sch], Corollary on p. 47). Explain the Mayer-Vietoris exact sequence ([Sch], p. 52) and prove its application to $(\mathbf{Z}/2, \mu_2)$-extensions ([Sch], bottom of p. 53; note that there is a mistake in Schoof's argument). Eventually, establish the existence of an exact sequence $0 \to \oplus\mathbf{Z}/2^k \to G \to \oplus\mu_{2^k} \to 0$ for $G/\mathbf{Z}$ a finite flat 2-power order group scheme ([Sch], Theorem on p. 53) and deduce that it implies Fontaine's Theorem over $\mathrm{Spec}\,\mathbf{Z}$—note the use of the Weil bound applied to $\mathscr{A}[2^n]$ to obtain a contradiction.

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