# KLEINE AG: ETALE HOMOTOPY THEORY

Organization:
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The goal of this Kleine AG is to establish a homotopy theory for schemes in analogy with the homotopy theory of topological spaces. In particular, we will see how to define the higher etale homotopy groups of schemes. Our main reference will be [AM].

**First talk. Simplicial sets.** (60 minutes) This talk should provide us with the necessary knowledge about simplicial sets and related notions from the general homotopy theory. One should introduce and give a reasonable understanding of the category of simplicial sets (and, more generally, the category of simplicial objects with values in a category $\mathscr{C}$) and its geometric realization via CW-complexes, the skeleton and coskeleton functors and their easiest properties. A final goal should be to explain the (pointed) extended homotopy category of simplicial sets, denoted by $\mathscr{K}$ (resp. $\mathscr{K}_0$) in [AM] §1. The references for this are [GZ], [AM] §1 and [GJ] Chap. 2. Proofs (e.g. of the geometric realization) should be omitted.

Also one should briefly recall the notion of pro-objects in a category and pro-representable functors (cf. [AM] Appendix §2).

**Second talk. Hypercoverings.** (45 minutes) In this talk you should explain hypercoverings, following [AM] §8. Recall briefly the definition of a (pointed) site. Define the notion of hypercovering $X.$ of a site $\mathscr{C}$ (8.4). Explain, why hypercoverings are 'better' than ordinary coverings (8.5). Introduce split simplicial objects (8.1) and split hypercoverings (8.8). Explain briefly Lemma 8.10, Proposition 8.11 and Corollary 8.12, which allow to lift maps resp. homotopies between truncated (split) hypercoverings to appropriately refined hypercoverings (you can decide, whether or how many details of the proofs you give, depending on your time management). Define the homotopy category $HR(\mathscr{C})$ of hypercoverings of $\mathscr{C}$, and deduce from the above that the dual category is filtering (Corollary 8.13). Depending on how much time is left, explain, without details, how one can compute the sheaf cohomology using hypercoverings, in particular, you can state Theorems 8.16 and 8.18.

**Third talk. Verdier functor and the etale homotopy type.** (30 minutes) Define the Verdier functor from pointed (locally connected) sites to pointed simplicial sets up to homotopy as in [AM] §9. Using the geometric realization functor one can now define the homology and homotopy pro-groups of a site $\mathscr{C}$ via the corresponding pointed CW-complex $\prod \mathscr{C}$. Using (8.16) from the last talk, deduce that cohomology with constant coefficients factors through the Verdier functor (9.3). Now, we are ready to define the etale homotopy type of schemes. State without proof that the etale site of a locally noetherian scheme $X$ is locally connected (9.4). Define the etale homotopy type of $X$ and its cohomology and higher homotopy groups (9.6),(9.7). Define also the homotopy pro-groups of a pointed locally connected topological space (9.8).

**Fourth talk. The fundamental group.** (30 minutes) The goal of this talk is to give an interpretation of the fundamental group $\pi_1(\mathscr{C})$ of a site in terms of isomorphism classes of locally trivial coverings of the final object [AM] (10.7). Locally trivial coverings correspond to descent data on hypercoverings. References for this talk are [Gr], [Mu](Chapter 7) [AM] §10.

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**Fifth talk. Pro-finiteness theorem.** (30 minutes) The goal of this talk is to explain Theorem 11.1 [AM], which states that the etale homotopy type of a connected geometrically unibranch noetherian scheme is pro-finite. In particular, you should explain pro-$\mathscr{C}$-completions of pro-objects in the category of pointed CW-complexes (or simplicial sets), where $\mathscr{C}$ is a class of groups [AM] (3.1)-(3.4). Give then as most details of the proof of Theorem 11.1 as possible [AM](11.2)-(11.6).

**Sixth talk. Comparison theorems.** (45 minutes) In this last talk we want to compare the usual cohomology and homotopy groups of a topological space resp. of a scheme with such defined via the Verdier functor. First, state (without proofs) Theorem 12.1 and its Corollary 12.2, saying that the two definitions of homotopy groups of a topological space $X$ coincide and that the homotopy type of the analytic space $X(\mathbb{C})$ attached to an algebraic $\mathbb{C}$-scheme $X$ is homotopic to the singular complex $S.X$ attached $X(\mathbb{C})$. Explain the generalization of Riemann's existence Theorem 12.9, saying that the pro-finite completions of the topological and the etale homotopy types of a given $\mathbb{C}$-scheme $X$ of finite type are isomorphic. State its corollaries (12.11)-(12.14).

## References


[AM] Artin M, Mazur B.: *Etale homotopy*, LNM **100**, Springer, 1969.

[GJ] Goerss P., Jardine J.: *Simplicial homotopy theory*, Progress in Math., Vol. **174**, Birkhauser, 1999.

[Gr] Grothendieck A.: *Technique de descente et theoremes d'existence en geometrie algebrique*, Sem. Bourbaki, exp. 190, 195, 1959-1960.

[GZ] : Gabriel P., Zisman M.: *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik, vol. **35**, Springer, 1967.

[Mu] Murre J.P.: *Lectures on An Introduction to Grothendieck's Theory of the Fundamental Group*