Kleine AG Algebraische Geometrie und Zahlentheorie

DESSINS D'ENFANTS

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In this "Kleine AG" we will dive into a topic which in a beautiful way connects 19th century complex analysis with 20th century arithmetic geometry, and very simple and intuitive objects with very abstract concepts.

The motivating question for the theory of dessins d'enfants (French for "children's drawings") is the following: given a compact Riemann surface X, it admits a unique structure of an algebraic curve over the complex numbers; when can this algebraic curve already be defined over a number field (or, equivalently, over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$)? In 1979, Belyĭ gave a surprisingly simple answer (see [Be79]): X is defined over a number field if and only if there is a nonconstant holomorphic map $X \longrightarrow \mathbb{P}^1(\mathbb{C})$ which is ramified over at most three points, which may (after applying a suitable Möbius transformation) be taken as the points 0, 1 and ∞. The "if" part had been known before by quite abstract arguments, whereas the "only if" part was new but with a completely elementary proof. Once this theorem is known, other criteria follow: X is defined over a number field if and only if it can be uniformised by a finite index subgroup in a cocompact triangle group, which in turn is equivalent to saying that it can be written as $\Gamma\backslash\overline{\mathbb{H}}$ where Γ is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. The precise meaning of these conditions will be explained in the seminar.

Belyĭ's theorem and its extensions have an interesting consequence: curves over number fields, which are very difficult to understand, can be described and defined by seemingly very simple objects. Namely the *algebraic* isomorphism class of a map $X \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$ as in Belyĭ's theorem is uniquely determined by the *holomorphic* isomorphism class of $X(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$, and this is in turn uniquely determined by its *topological* structure, and also by the topological structure of the induced unramified covering of $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$. So every algebraic curve over $\overline{\mathbb{Q}}$ can be defined by giving a finite cover of a sphere minus three points. It also can be defined by giving a certain type of graph embedded into the topological surface $X(\mathbb{C})$ (i.e., forgetting the complex structure) that determines how $X(\mathbb{C})$ is "wrapped" around the sphere $\mathbb{P}^1(\mathbb{C})$. Such graphs have been named "dessins d'enfants" by Grothendieck. And, as already mentioned, it can be defined by giving a finite index subgroup of one of the most classical groups in number theory or hyperbolic geometry. As all these different data correspond to algebro-geometric data over number fields, their isomorphism classes are acted upon by the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, one of the most important and at the same time most mysterious groups in mathematics. Belyĭ's theorem also implies that these actions are faithful, so the absolute Galois group is embedded in the automorphism groups of some very concrete objects. It is still a widely open task to understand these actions and these embeddings.

Grothendieck, who initiated the study of dessins d'enfants in his famous paper *Esquisse d'un programme* [Gr86], was fascinated by the idea that such simple objects embody such deep mathematics. In loc. cit., p. 12 he writes:

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> Cette découverte, qui techniquement se réduit à si peu de choses, a fait sur moi une impression très forte, et elle représente un tournant décisif dans le cours de mes réflexions, un déplacement notamment de mon centre d'intérêt en mathématique, qui soudain s'est trouvé fortement localisé. Je ne crois pas qu'un fait mathématique m'ait jamais autant frappé que celui-là, et ait eu un impact psychologique comparable. (Je puis faire exception pourtant d'un autre "fait", du temps où, vers l'âge de douze ans, j'étais interné au camp de concentration [...]. C'est là que j'ai appris, par une détenue, Maria, qui me donnait des leçons particulières bénévoles, la définition du cercle. Celle-ci m'avait impressionné par sa simplicité et son évidence, alors que la propriété de "rotondité parfaite" du cercle m'apparaissait auparavant comme une réalité mystérieuse au-delà des mots. [...]) Cela tient sûrement à la nature tellement familière, non technique, des objets considérés, dont tout dessin d'enfant griffonné sur un bout de papier (pour peu que le graphisme soit d'un seul tenant) donne un exemple parfaitement explicite. A un tel dessin se trouvent associés des invariants arithmétiques subtils, qui seront chamboulés complètement dès quon y rajoute un trait de plus.[3]

We hope to convey in this seminar at least a little of this fascination. It goes without saying that due to the many facets of the topic — algebraic geometry, number theory, graph theory, topology, classical and differential geometry, group theory, ... — one can just present a tiny piece of the whole picture in a one-day seminar; many exciting developments have to remain completely untouched.

The prerequisites are different than usual: techniques of modern algebraic geometry are only needed in the first talk (and there also only in a very mild form) to prove Belyĭ's Theorem, the rest is, as Grothendieck said, technically simple. A great part of the programme deals with combinatorics, complex analysis and geometry, but on an elementary level, so you need not be a combinatorist, complex analyst or geometer to understand or give these talks.

The Programme

First talk: Riemann surfaces and Belyĭ's Theorem (45 minutes). The aim of this talk is to prove Belyĭ's Theorem in its original formulation and derive some easy consequences.

First the proof of Belyĭ's Theorem should be explained along the lines of [BoHu00, §1, 2 and 4]. However their presentation only uses the language of algebraic geometry, and the referent should translate Belyĭ's Theorem into an analytic statement about compact Riemann surfaces, because this is the formulation that leads to all the geometric and topological interpretations. Also the terminology of [Schn94b, I.§1, Definition 1] should be introduced (pre-clean and clean Belyĭ morphisms and Belyĭ pairs) and the Corollary in [Schn94b, p. 50] should be stated.

[3]This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. (With the exception of another "fact", at the time when, around the age of twelve, I was interned in the concentration camp [...]. It is there that I learnt, from another prisoner, Maria, who gave me free private lessons, the definition of the circle. It impressed me by its simplicity and its evidence, whereas the property of "perfect rotundity" of the circle previously had appeared to me as a reality mysterious beyond words. [...]) This is surely because of the very familiar, non-technical nature of the objects considered, of which any childs drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin, we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke. — Translation by Leila Schneps, [LoSchn97]

Then the referent should include a reminder of some classical results about ramified coverings of Riemann surfaces, as e.g. presented in [La09, sect. 1.4, 4.6, 4.8]: finite holomorphic maps are ramified coverings; for $S \subseteq X$ discrete, every finite topological covering $Y_0 \longrightarrow X \smallsetminus S$ bears a canonical holomorphic structure and can uniquely be extended to a ramified covering $Y \longrightarrow X$ of Riemann surfaces; signatures and universal ramified coverings with prescribed signature. Theorem 4.8.3 in loc. cit. This opens the way to a purely topological description of Belyĭ functions which is undertaken in the second talk.

Second talk: Dessins d'enfants (45 minutes). This talk explores the combinatorial and topological interpretations of Belyĭ's Theorem. Since a Belyĭ pair $(X, \beta : X \longrightarrow \mathbb{P}^1(\mathbb{C}))$ is uniquely determined by the underlying topological data, the aim is to codify these topological data as simply as possible. As to the topological and combinatorial background needed, we suggest to follow [LaZv04, 1.1.1—1.1.5]. This may seem a lot, but it is not, since this part of the book is very easy to read and introduces some very basic notions like fundamental groups and surfaces, which we of course assume to be known. Therefore the discussion should be restricted to the following: constellations (Def. 1.1.1), the cartographic group (Def. 1.1.2), the correspondence between constellations and ramified coverings of S^2 (section 1.2.3), maps (Def. 1.3.6), isomorphisms of maps (Def. 1.3.7 — subtle notion, see the remarks after that definition), hypermaps (section 1.5.1), the relation between hypermaps and triangulations (section 1.5.4).

Then these considerations should be connected to Belyĭ's Theorem. The content to be covered is in [Schn94b, II.§2—4]. Please do *not* use Schneps' definitions of dessins and (pre-) clean[4] dessins because they are very prone to misunderstanding. What Schneps means by Definition 2 is Lando and Zvonkin's hypermaps, and a clean dessin is then simply a map in the sense of [LaZv04]. The notion of a pre-clean dessin is obtained by allowing "open edges" in the definition of a map. Using these definitions, loc. cit. becomes perfectly sensible and understandable. Of importance are §3 and Schneps' definition of the (abstract, oriented) cartographical group, the rest is already contained in [LaZv04].

Finally some remarks should be made about the Galois operation on (isomorphism) classes of dessins, we suggest [Wo06, Prop. 7 (page 23)] without proof, [Schn94b, Prop. II.1] with proof and [Schn94b, Theorem II.4] without proof. Define the field of moduli of a dessin (see [LaZv04, sect. 2.4.1.2], quote (and draw) a few of the easier examples in [LaZv04].

Third talk: Uniformisation (45 minutes). This talk develops a more geometric view on Belyĭ's Theorem. The goal of this talk is to state and understand [Si01, Thm. 4.1] and to indicate some ideas of its proof. Some of the involved concepts (namely triangle groups) are straightforwardly geometric, and thus here the differential geometry of Riemann surfaces comes into play. To explain the interplay between complex structures and metrics in general would consume too much time and hence we advise the referent only to refer to those aspects which are strictly necessary to understand the theorem.

To begin with, one should talk about the three simply connected Riemann surfaces $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{C})$ and their automorphism groups. If X is a compact surface of genus g with n points removed, then the isomorphism type of universal covering space of X is determined by the pair (g, n).

Then discontinuous groups of automorphisms should be recalled, in particular the definition (see e.g. [La09, 4.3]) and the characterisation of discontinuous groups on $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{C})$, i.e. [Ma74, II. Thm. 2.3]. Discontinuous groups on the Riemann sphere and the complex plane can easily be classified: on the sphere, every such group is finite and, up to conjugacy, contained in

[4] In fact, in [Schn94b] the notion of a clean dessin is not even properly introduced, although used

the group of rotations of the sphere $\mathrm{SO}(3) \subset \mathrm{PGL}_2(\mathbb{C})$. Present the list of these groups up to conjugacy: cyclic, dihedral, tetrahedral, octahedral and icosahedral, see [JoSi87, sect. 2.13] or [La09, sect. 4.2]. Discontinuous groups on the complex plane are also very few, see e.g. [La09, sect. 2.6] and list them.

The hyperbolic case is more complicated. The three classes of nontrivial automorphisms of $\mathbb{H}$ (elliptic, parabolic and hyperbolic ones) should be defined, with Prop. 1.13, Prop. 1.16, Prop. 1.17 and Prop. 1.18 in [Shi71]. The referent should then follow [Shi71, 1.3 and 1.5] to explain the following construction: if $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a discrete subgroup, denote by $\overline{\mathbb{H}} \subseteq \mathbb{H} \cup \mathbb{P}^1(\mathbb{R})$ the union of $\mathbb{H}$ with the cusps of Γ. Then a topology on $\overline{\mathbb{H}}$ and a Riemann surface structure on $\Gamma\backslash\overline{\mathbb{H}}$ are constructed. It is important to recall this quite explicitly because it must not be confused with the orbifold or stack quotient.

Then the referent should outline [Ma74, sect. II.3—5] (triangle groups) and draw some pictures, e.g. reproduce those in loc.cit. Important examples of triangle groups are given by the modular groups $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}) = T(2,3,\infty)$ and $\Gamma(2) = T(\infty,\infty,\infty)$. As to these examples, please resume the following from [Ma74, chap. III]: Thm. 3.1, Thm. 3.2, Cor. 3.2, Example 1. One also has $\Gamma_0(2) = T(2,\infty,\infty)$ which is important because $\Gamma_0(2)$ can be viewed as the geometric incarnation of the oriented cartographic group, see [JoSt97, §5] which also contains a combinatorial interpretation of $\Gamma(2)$.

After all this has been done the referent should discuss [Si01, Thm. 4.1]. However the presentation in loc. cit. is fairly short, and so she may instead consult [JoSi96, sect. 4].

Fourth talk: Platonic, quasi-platonic and modular surfaces (45 minutes). This talk discusses some particularly rich examples.

Introduce the principal congruence subgroup $\Gamma(N) = \ker(\mathrm{PSL}_2(\mathbb{Z}) \longrightarrow \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}))$ and the modular curves $X(N) = \Gamma(N)\backslash\overline{\mathbb{H}}$. Discuss the interpretation of $X(N)$ as parametrizing complex elliptic curves with level-N-structures (see e.g. [Hu04, 11.§2—§3]). Discuss [La09, sect. 5.3—5.5 and 5.7]. This comprises basically the special examples for $N \leq 6$, the underlying classical function theory (the j-function and the λ-function) and the fact that the forgetful morphism $X(N) \longrightarrow X(1) = \mathbb{P}^1$ is a clean Belyĭ morphism (this is not explicitly said in loc. cit. but follows directly). It is helpful also to look at [ShaVo90, pp. 212—213] which brings the connection to finite automorphism groups of the sphere and platonic solids. Then jump to [La09, sect. 11.6] where $X(7)$ and Klein's 14-gon are discussed. This is a really beautiful object with many very special properties. If time permits, it would be nice to mention some of them; the referent may look into the collection [Le99] (primarily the articles [KaWe99], [Macb99] and [El99, sect. 2.1—2.2]) and [Ma74], pp. 94—95, 102—104, 115—116, and talk about what she finds most interesting.

Then the referent should resume [Si01, sect. 6] about platonic[5], quasi-platonic and smooth Belyĭ surfaces. Caution: the notion of a "smooth Belyĭ surface" is somewhat unfortunate since all Riemann surfaces are smooth, but this is not meant. In the last sentence of subsection 6.1 it must be "smooth Belyĭ surfaces" instead of "Belyĭ surfaces". Then summarise [Wo06, sect. 4.6] (quasi-platonic curves and curves with many automorphisms).

Fifth talk: Relations to Diophantine equations (45 minutes). In this talk four connections between dessins and Diophantine equations are presented. Three of them are set up by taking a difficult problem about integers, formulating an analogous problem for polynomials and translating it into the theory of dessins; the fourth one is a consideration of the Fermat

[5] An equivalent definition of platonicity is in [KaWe99, sect. 2]; this is closer to intuition and geometry

curves $X^n + Y^n = Z^n$ from a point of view of dessins. Summarise the following sections of [LaZv04]:

2.5.1 A Bound of Davenport-Stothers-Zannier. — The corresponding number theory problem, which is not mentioned in [LaZv04], is Marshall Hall's Conjecture, see e.g. the introduction of [El00].

2.5.3 The Fermat Curve.

2.5.4 The *abc* Conjecture.

2.5.6 Pell Equation for Polynomials. — As to the Pell Equation in number theory see e.g. [MaPa00, pp. 22, 50—54].

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