Geometry and Arithmetic of del Pezzo Surfaces

A **del Pezzo surface** is a smooth projective geometrically integral surface X over a field k with ample anticanonical bundle $-K_X$. Thus, del Pezzo surfaces are exactly the Fano surfaces. The **degree** of a del Pezzo surface is $d := (-K_X, -K_X) = (K_X, K_X)$.

Talk 1: Geometric classification (45 minutes)

Let X be a del Pezzo surface over a separably closed field $k = k^{\mathrm{sep}}$. Then $1 \leq d \leq 9$, and $X \cong \mathbf{P}^1_k \times_k \mathbf{P}^1_k$ (in which case $d = 8$) or X is the blow-up of $\mathbf{P}^2_k$ in $0 \leq r \leq 8$ closed k-points in general position (in which case $d = 9 - r$) [Manin], p. 119, Theorem 24.4. Here, "general position" means that no 3 points lie on a line, no 6 points lie on a conic, and no 8 points lie on a singular cubic, with one of the points at the singularity. In particular, all del Pezzo surfaces are rational. Birational invariance of being a del Pezzo surface [Manin], p. 125, Corollary 24.5.2.

Talk 2: Geometric properties of del Pezzo surfaces (45 minutes)

The Picard group of X is isomorphic to $\mathbf{Z}^{10-d}$ [Manin], p. 118, Lemma 24.3.1; p. 126, Section 25; every irreducible curve with negative self-intersection number on X is exceptional [Manin], p. 118, Theorem 24.3 (ii); root systems [AVA1], p. 6.

Talk 3: The anticanonical model (15 minutes)

The anticanonical map $X \to \mathbf{P}^d_k$ is a closed immersion for $d \geq 3$, so X can be realised as a degree-d surfaces in $\mathbf{P}^d_k$ [AVA1], p. 7 f., 1.5.4 and [Corn], p. 4 f., Proposition 2.1 (f).

Talk 4: Arithmetic of del Pezzo surfaces: Hasse principle and weak approximation (60 minutes)

Let k be a global field. Cover the cases $d = 5, 6, 7, 8, 9$ [AVA1], p. 8–11: These satisfy the Hasse principle, weak approximation, and, if $X(k) \neq \emptyset$, they are birational to $\mathbf{P}^2_k$, e. g. if $d = 9$, $X_{k^{\mathrm{sep}}} \cong \mathbf{P}^2_{k^{\mathrm{sep}}}$, so X is a Severi-Brauer surface, which satisfies the Hasse principle by the theorem of Albert-Brauer-Hasse-Noether.

Shortly mention the cases $d = 1, 2, 3, 4$, [Poonen], p. 218.

Talk 5: Arithmetic of del Pezzo surfaces: The Brauer group (45 minutes)

Prove the Picard-Brauer sequence from the Hochschild-Serre spectral sequence $\mathrm{H}^p(G_k, \mathrm{H}^q_{\text{ét}}(X_{k^{\mathrm{sep}}}, \mathbf{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbf{G}_m)$

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^{\mathrm{sep}}})^{G_k} \to \mathrm{Br}(k) \to \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{k^{\mathrm{sep}}})) \to \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}(X_{k^{\mathrm{sep}}})) \to \mathrm{H}^3_{\text{ét}}(k, \mathbf{G}_m)$$

[AVA2], p. 9. Conclude $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}(X_{k^{\mathrm{sep}}}))$ for del Pezzo surfaces over global fields. Discuss the Galois action on the Picard group [AVA1], p. 7, 1.5.3. Calculate $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{H}^1_{\text{ét}}(k, \mathrm{Pic}(X_{k^{\mathrm{sep}}}))$ [Corn], p. 6 for k a global field.

References

[AVA1] Anthony Várilly-Alvarado, *Arithmetic of del Pezzo surfaces*, `http://math.rice.edu/~av15/Files/LeidenLectures.pdf`.

[AVA2] Anthony Várilly-Alvarado, *Arithmetic of del Pezzo and K3 surfaces*, `http://rational.epfl.ch/dPK3.pdf`.

[Corn] Patrick Corn, *The Brauer-Manin obstruction on Del Pezzo surfaces of degree* 2, `http://www.mathcs.emory.edu/~pcorn/paper2corr.pdf`.

[Manin] Yuri Manin, *Cubic Forms*, 1974.

[Poonen] Bjorn Poonen, *Rational points on varieties*, http://www-math.mit.edu/~poonen/papers/Qpoints.pdf.