Kleine AG: Deligne-Lusztig theory

Organization:
Timo Keller[1]
Stephan Neupert[2]

Let G be a connected, reductive group over a finite field $\mathbb{F}_q$. Then $G(\mathbb{F}_q)$ is a finite (abstract) group and we may ask for (all) irreducible representations (in $\mathbb{C}$-vector spaces) of this group. To answer this question, Deligne and Lusztig gave in 1976 the following construction:
First fix a maximal torus T and a Borel B inside G. Let F be the q-Frobenius on G and assume it fixes T (i.e. T is already defined over $\mathbb{F}_q$). To any element $w \in W(G)$ one then associated a finite Galois-cover $Y_{\dot{w}}$ over the Schubert cell $BwB/B \subset G/B$ in the flag variety. This space $Y_{\dot{w}}$ admits a canonical action of $G(\mathbb{F}_q) = G^F$ and of T^{F_w} (where F_w is a twisted version of the Frobenius). Thus by passing to ℓ-adic cohomology one obtains a virtual G^F-representation (a priori in $\overline{\mathbb{Q}}_\ell$-vector spaces, but one may just fix some (non-canonical) isomorphism to $\mathbb{C}$)

$$H_c^*(Y_{\dot{w}}, \overline{\mathbb{Q}}_\ell) = \sum\nolimits_i (-1)^i H_c^i(Y_{\dot{w}}, \overline{\mathbb{Q}}_\ell)$$

Finally one may decompose this representation according to characters θ of T^{F_w} to get smaller virtual G^F-representation R_w^θ.
The main theorems of Deligne-Lusztig theory now assert, that any irreducible G^F-representation can be found in one of the R_w^θ and the representations R_w^θ are quite often irreducible themselves.

The details of several of the proofs are rather long and technical (though elementary) and thus not terribly suitable for talks in a Kleine AG. So speakers are encouraged to replace such proofs either by sketches of the main ideas or to illustrate them with explicit examples (e.g. for $G = GL_2, GL_3$).
On request the second organizer (S.N.) may provide you with seminar notes (containing even some explicit examples) or further explanations.

1. Geometric background (45 minutes)

Explain how varieties with an action of a finite group give representations in the ℓ-adic cohomology with compact support. Define Lefschetz numbers and provide some general properties. Make sure to mention the Lefschetz fix-point formula, which is of independent interest. Finally provide the formula

$$\operatorname{Tr}\left(g, H_c^*(X, \overline{\mathbb{Q}}_\ell)\right) = \operatorname{Tr}\left(u, H_c^*(X^s, \overline{\mathbb{Q}}_\ell)\right) \qquad (*)$$

where $g = s \cdot u$, s and u are powers of g with $gcd(p, ord(s)) = 1$, $ord(u) \in p^{\mathbb{N}}$ and X^s denotes the fixed points of X under s. If time permits state the variant [Ca, proposition 7.2.5] (or in a slightly different form [DL, corollary 3.10]). Please contact the speaker of talk 4 to ensure that the formulation of these last formulas fit together.
An overview without proofs can be found in [Ca, 7.1]. For more details (especially proofs) check out [Lu, section 1]. The formula $(*)$ is theorem 3.2 in [DL] (where one can find most proofs as well, but scattered throughout the article).

2. Definition of DL-varieties (30 minutes)

Define Deligne-Lusztig varieties X_w and $Y_{\dot{w}}$[3] and the representations R_w^θ (depending on a character θ on T^{F_w}). It would be an excellent idea to discuss the geometry in at least one explicit example, e.g. $G = GL_2$ and w the non-trivial element. In this case one simply gets

$$Y_{\dot{w}} = \{(x_0, x_1) \in \mathbb{A}^2 \mid (x_0 x_1^q - x_0^q x_1)^{q-1} = 1\} \to X_w = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q).$$

A very elegant definition of the spaces is given in [DL, 1.4 and 1.7-1.9]. However the presentation in [Yo, 2.1-2.3] may feel cleaner.

[1] <Vorname>.<Nachname>@mathematik.uni-regensburg.de
[2] <Nachname>@ma.tum.de
[3] X_w is called $X(w)$ and $Y_{\dot{w}}$ is called $\widetilde{X}(\dot{w})$ in [DL]

3. Orthogonality theorem (45 minutes)

Briefly explain the definition of $X_{T\subset B}$, $Y_{T\subset B}$ and the representations $R^{\theta}_{T\subset B}$ and state that they are equal to some of the $R_w^{\theta\circ ad\dot{w}}$. Note that here θ is a character of T^F contrary to talk 2.
The main focus should then lie on the orthogonality theorem

$$\langle R^{\theta}_{T\subset B}, R^{\theta'}_{T'\subset B'}\rangle = \left|\{w \in W(T,T')^F \mid ad(w)^*\theta = \theta'\}\right|$$

which is crucial for most further computations. Finally conclude that $\pm R^{\theta}_w$ is irreducible if and only if θ is in general position, i.e. θ is not fixed by any non-trivial element in the Weyl group W.
For the definition of $X_{T\subset B}$, ... see [DL, 1.17-1.20] and [Yo, 2.4]. Note that [Ca, 7.2] and [Lu, 2.1-2.2] use slightly different spaces (namely affine fibrations over $Y_{T\subset B}$) which nevertheless give the same representations $R^{\theta}_{T\subset B}$. Proofs of the orthogonality theorem can be found in [DL, section 6] and slightly simplified ones in [Lu, theorem 2.3] and [Ca, 7.3]. For irreducibility results see [Lu, corollary 2.5] or [Ca, 7.3.5].

4. A character formula (45 minutes)

State first, that $R^{\theta}_{T\subset B}$ is actually independent of B (cf. e.g. [Lu, corollary 2.4], but see the comment after definition 3.6 in [Yo]) and is therefore abbreviated by R^{θ}_T from now on. The main part of this talk is the character formula

$$R^{\theta}_T(g) \coloneqq \mathrm{Tr}(g, R^{\theta}_T) = |C^0_G(s)^F|^{-1} \cdot \sum_{\substack{x\in G^F\\ xsx^{-1}\in T^F}} \theta(xsx^{-1}) Q^{C^0_G(s)}_{x^{-1}Tx}(u)$$

where $g = s\cdot u$ as in the first talk and Q denotes the Green function (cf. e.g. [DL, definition 4.1]). The importance of this formula lies in the fact, that it reduces computations of traces to the case of unipotent elements and trivial characters for θ.
Explicit examples may help to get a better understanding of this formula. While the details of its proof seem not to be of great importance, explain at least the connection to the formula $(*)$ and the appearance of the centralizer $C^0_G(s)$.
The best reference seems to be [Ca, theorem 7.2.8], but you might wish to look at [DL, theorem 4.2] as well. The example GL_2 is worked out in [DM, section 15.9].

5. Irreducible representations in DL-representations (60 minutes)

Prove that any irreducible representation occurs in some DL-representation. To this end one needs to express the trivial and the regular representation via DL-representations and compute the dimension of R^{θ}_T.
One reference is [Lu, 2.7, 2.9, 2.11 and 2.12]. Alternatively see [Ca, 7.4.2, 7.5.1, 7.5.6, 7.5.8] for a proof relying heavily on the computation of traces of semi-simple elements or [DM, 12.13, 12.9, 12.14], which provides as well some slight generalizations. It is not recommended to follow [DL] in this talk, even though the results are already present there.

References

[Ca] R. Carter, Finite groups of Lie type. Conjugacy classes and complex characters. Reprint of the 1985 original. Wiley Classics Library. A Wiley-Interscience Publication, xii+544 pp.

[DL] P. Deligne, G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. **103** (1976), no. 1, pp. 103-161, `http://www.jstor.org/stable/pdf/1971021.pdf?acceptTC=true`

[DM] F. Digne, J. Michel, Representations of Finite Groups of Lie Type, London Math. Soc. Student Texts **21**. Cambridge Univ. Press, 1991, iv+159 pp.
Errata at `http://webusers.imj-prg.fr/~jean.michel/papiers/errata.pdf`.

[Lu] G. Lusztig, Representations of finite Chevalley groups. Expository lectures from the CBMS Regional Conference held at Madison, Wis. August 12, 1977. CMBS Regional Conference Series in Mathematics, 39. Am. Math. Soc., Providence, R.I. v+48 pp.
In case you cannot find this article, please contact one of the organizers to provide you with a scan.

[Yo] T. Yoshida, An introduction to Deligne-Lusztig theory, `https://www.dpmms.cam.ac.uk/~ty245/Yoshida_2003_introDL.pdf`